# Structural stability for variable exponent elliptic problems. I. The $p(x)$-laplacian kind problems.

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# Structural stability for variable exponent elliptic problems. I. The $p(x)$-laplacian kind problems.

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**Abstract**

We study the structural stability (i.e., the continuous dependence on coefficients) of solutions of the elliptic problems under the form

$$b(u_n) - \operatorname{div} \mathfrak{a}_n(x,\nabla u_n) = f_n.$$

The equation is set in a bounded domain $\Omega$ of $\mathbb{R}^N$ and supplied with the homogeneous Dirichlet boundary condition on $\partial\Omega$. Here $b$ is a non-decreasing function on $\mathbb{R}$, and $\big(\mathfrak{a}_n(x,\xi)\big)_n$ is a family of applications which verifies the classical Leray-Lions hypotheses but with a variable summability exponent $p_n(x)$, $1 < p_- \le p_n(\cdot) \le p_+ < +\infty$. The need for making vary $p(x)$ arises, for instance, in the numerical analysis of the $p(x)-$laplacian problem. Uniqueness and existence for these problems are well understood by now. We apply the stability properties to further generalize the existence results.

The continuous dependence result we prove is valid for weak and for renormalized solutions. Notice that, besides the interest of its own, the renormalized solutions' framework also permits to deduce optimal convergence results for the weak solutions.

Our technique avoids the use of a fixed duality framework (like the $W_0^{1,p(x)}(\Omega)$—$W^{-1,p'(x)}(\Omega)$ duality), and thus it is suitable for the study of problems where the summability exponent $p$ also depends on the unknown solution itself, in a local or in a non-local way. The sequel of this paper will be concerned with well-posedness of some $p(u)$-laplacian kind problems and with existence of solutions to elliptic systems with variable, solution-dependent growth exponent.

*Key words:* $p(x)$-laplacian, Leray-Lions operator, variable exponent, thermo-rheological fluids, well-posedness, continuous dependence, convergence of minimizers, Young measures
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## 1. Introduction

The purpose of this paper is to present a technique for dealing with sequences of solutions of degenerate elliptic problems with variable coercivity and growth exponents $p$. The prototype equations are $-\mathrm{div}\left(|\nabla u|^{p(x)-2}\,\nabla u\right) = f$ (known as the $p(x)$-laplacian equation) and $-\mathrm{div}\left(|\nabla u|^{p(u)-2}\,\nabla u\right) = f$ (which we call the $p(u)$-laplacian). Issues related to the passage-to-the-limit techniques are:

– existence of solutions;
– study of convergence of various approximations,
  including the numerical analysis of these problems.

By "variable exponent $p$", we mean $p$ that can depend explicitly on the space variable $x$ and on the approximation parameter $n$. In the sequel [6] of this paper we also allow for the dependence of $p$ on the unknown solution $u_n$. From the structural stability theory we will derive new existence results (including those for $p(u)$-laplacian kind problems). A uniqueness analysis for the $p(u)$-laplacian will also be carried out in [6].

Problems with variable exponents $p(x)$ and $p_n(x)$ were arisen and studied by Zhikov in the pioneering paper [63] and a series of subsequent works including [64, 66, 65, 2, 68, 69]. In what concerns the passage-to-the-limit techniques, Zhikov's methods include semicontinuity arguments and an ingenious adaptation of the classical Minty-Browder monotonicity argument; see in particular [68, Lemmas 8,9]. Similar approaches were used by Haehnle and Prohl [37] and by Wróblewska (see [62] and references therein). Our argument is longer but more straightforward. Its main ingredient is the convergence analysis in terms of Young measures associated with a weakly convergent sequence of gradients of solutions, as presented by Dolzmann, Hungerbühler and Müller (see [27, 41] and references therein; see also Gwiazda and Świerczewska-Gwiazda [36]).

Let us state the model problem for our study. Let $\Omega$ be a bounded domain of $\mathbb{R}^N$ with Lipschitz boundary. We deal with nonlinear elliptic equations in $\Omega$ under the general form

$$b(u) - \mathrm{div}\, \mathfrak{a}(x,u,\nabla u) = f, \tag{1}$$

where $b:\mathbb{R}\longrightarrow\mathbb{R}$ is nondecreasing, normalized by $b(0)=0$. Further, we assume that $\mathfrak{a}:\Omega\times(\mathbb{R}\times\mathbb{R}^N)\longrightarrow\mathbb{R}^N$ is a Carathéodory function with

$$\mathfrak{a}(x,z,0)=0 \text{ for all } z\in\mathbb{R} \text{ and a.e. } x\in\Omega \tag{2}$$

satisfying, for a.e. $x\in\Omega$, for all $z\in\mathbb{R}$, the strict monotonicity assumption

$$(\mathfrak{a}(x,z,\xi)-\mathfrak{a}(x,z,\eta))\cdot(\xi-\eta)>0 \qquad \text{for all } \xi,\eta\in\mathbb{R}^N,\ \xi\neq\eta\ . \tag{3}$$

Typically, $\mathfrak{a}$ is assumed to satisfy the following growth and coercivity assumptions[1] in $\nabla u$ with variable exponent $p$ depending both on $x$ and on the unknown values $u(x)$:

$$|\mathfrak{a}(x,z,\xi)|^{p'(x,z)} \leq C\left(|\xi|^{p(x,z)} + \mathcal{M}(x)\right), \tag{4}$$

$$\mathfrak{a}(x,z,\xi)\cdot\xi \geq \frac{1}{C}\,|\xi|^{p(x,z)}. \tag{5}$$

Here $C$ is some positive constant, $\mathcal{M} \in L^1(\Omega)$,

$$p : \Omega\times\mathbb{R} \longrightarrow [p_-, p_+] \text{ is Carathéodory, } 1 < p_- \leq p_+ < +\infty, \tag{6}$$

and $p'(x,z) := \frac{p(x,z)}{p(x,z)-1}$ is the conjugate exponent of $p(x,z)$. Note that more general than (4),(5) $x$-dependent growth and coercivity conditions of the Orlicz type for the nonlinearity $\mathfrak{a}$ can be considered. For the $x$-independent case, see Kačur [42] and a series of works of Benkirane and al. (see e.g. [13]); for the $x$-dependent case, we refer to the works of Gwiazda, Świerczewska-Gwiazda and Wróblewska (see [36, 62] and references therein). Also note that the technique of Young measures we use actually applies to monotone systems of equations, under a large variety of monotonicity assumptions replacing (3) (see Hungerbühler [41]; cf. Wróblewska [62]).

For the sake of simplicity, we supplement (1) with the homogeneous Dirichlet boundary condition:

$$u = 0 \qquad \text{on } \partial\Omega. \tag{7}$$

In this paper, we mainly limit ourselves to the case of a source term $f$ which is at least in $L^1(\Omega)$. We do not treat the case of source terms which are general Radon measures; for elliptic problems with a constant exponent $p$ in (4),(5) and measure source terms, we refer to [15, 24, 47, 46] for the existence and stability results.

The case of the $p(x)$-laplacian kind problems (the $p(x)$-laplacian operator $\Delta_{p(x)}u$ corresponds to the choice $\mathfrak{a}(x,u,\nabla u) := |\nabla u|^{p(x)-2}\nabla u$) has been extensively studied in the last decades; see e.g. [63, 64, 1, 33, 23, 22, 9, 38, 35, 68]. The interest for this study was boosted by the introduction of the $p(x)$-laplacian into models of electrorheological and thermorheological fluids (see in particular Rů žička [55, 56], Rajagopal and Růžička [54], Diening [25], Zhikov [65, 66], Antontsev and Rodrigues [8], Gwiazda and Świerczewska-Gwiazda [36]), and more recently, in the context of image processing (see Chen, Levine and Rao [21]; cf. Harjulehto, Hästö and Latvala [39]). Let us stress that in general, the nonlinearity rate $(p-2)$ may depend not only on $x \in \Omega$, but also on parameters affected by the values of the unknown solution $u$ itself.

When the dependency of $p$ on $u$ is local, this assumption leads to the problems of the kind (1)-(6). Note that a related, although far more complicated,

[1] The form (4),(5) is taken for the sake of simplicity; in particular, for the existence result of Theorem 3.11, we will take more general growth and coercivity assumptions.

minimization problem with $p = p(\nabla u)$ was suggested in [14]. A much more practical case is the one of coupled problems, where the exponent $p$ in (1) depends on $x$ through a solution $v$ of a PDE coupled with (1). Examples of such problems are given in [65, 66, 8, 68, 69]. We will study both local $p(u)$ and non-local $p[u]$-laplacian kind problems in the sequel [6] of this paper.

Analysis of the $p(x)$-laplacian kind problems (1),(7) with $u$-independent exponent $p(x)$ requires good understanding of the variable exponent Lebesgue and Sobolev spaces $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$. The studies carried out before 1990 include the pioneering works by Orlicz, Nakano, Hudzik, Musielak, Tsenov, Sharapudinov and other authors. In the last twenty years, many new works were devoted to this subject. For information on the variable exponent Lebesgue and Sobolev spaces $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$ we refer to [49, 43, 29, 38], to the surveys [32, 9, 26] and references therein. Let us only mention here that conditions on $p(x)$ have been found under which these spaces have properties similar to the ones of the classical Lebesgue and Sobolev spaces. Roughly speaking, $L^{p(\cdot)}(\Omega)$ possesses many of the important properties of the usual $L^p$ spaces, $1 < p = const < +\infty$, under the sole assumption that $p(\cdot)$ is measurable on $\Omega$ and for a.e. $x \in \Omega$, the value $p(x)$ belongs to some interval $[p_-, p_+] \subset (1, +\infty)$. The situation with the generalized Sobolev spaces $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ is much more involved. The key properties such as the optimal Sobolev embedding into $L^{p^*(x)}(\Omega)$, convergence of mollifiers' regularizations, density of the smooth functions, translation estimates, identification of $W_0^{1,p(\cdot)}(\Omega)$ with $W_0^{1,1}(\Omega) \cap W^{1,p(\cdot)}(\Omega)$, require additional assumptions; the most practical one is the so-called log-Hölder continuity assumption on $p(\cdot)$ (see (11) below) due to Fan and Zhikov.

The homogeneous Dirichlet condition (7) can be interpreted in different ways. When (1) can be seen as the Euler-Lagrange equation for some variational problem, minimization over any closed linear space included between $W_0^{1,1}(\Omega) \cap W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ leads to a different notion of solution (Zhikov [63]; see also [64, 65, 2]). All these notions of solution coincide when $\Omega$ is a Lipschitz domain and the exponent $p$ is log-Hölder continuous.

As soon as the crucial properties of the chosen solution space (e.g., $W_0^{1,p(\cdot)}(\Omega)$) are established, the $p(x)$-laplacian kind problems can be studied by the variational techniques or, more generally, by the classical Leray-Lions approach (see [44, 45]). In this way, well-posedness in $W_0^{1,p(\cdot)}(\Omega)$ for the problems of the kind (1)-(6) with $u$-independent diffusion flux $\mathfrak{a}$ was established. Without being exhaustive, we refer to the papers [63, 64, 1, 33, 23, 22, 9, 38, 35, 68] and references therein for existence and uniqueness results for weak solutions of the problem with a source term $f$ in the spaces $L^{p'(\cdot)}(\Omega), L^{(p^*(\cdot))'}(\Omega)$ or, most generally, in the dual space $W^{-1,p'(x)}(\Omega)$ of $W_0^{1,p(\cdot)}(\Omega)$. For source terms $f \in L^1(\Omega)$, the notions of entropy solutions (see [12, 20]) and renormalized solutions (see [16, 48]) of nonlinear elliptic problems have been successfully adapted in the works [59, 11, 51, 61].

In this paper, we first concentrate on the question of continuous dependence on a parameter $n$ of solutions of the $p_n(x)$-laplacian kind equations. Such structural stability results are useful, in particular, for the study of convergence of numerical approximations of the $p(x)$-laplacian. Indeed, it is necessary, for such a numerical study, to approach $p(x)$ by some piecewise constant or piecewise polynomial functions $p_h(x)$, $h$ being the discretization parameter (see e.g. Haehnle and Prohl [37] and the forthcoming paper [7] for numerical approximation of problems involving $p(x)$-laplacian).

The question of structural stability, i.e. the dependency of solutions on the operator $\mathfrak{a}_n$, is well studied in the case the underlying PDEs are the Euler-Lagrange equations associated with convex functionals $J_n$. Then structural stability stems from the $\Gamma$-convergence of $J_n$ to a limit $J$ (see e.g. [70]). This variational approach was also extended to the variable exponent framework (see in particular [63, 4]).

In the case $p \equiv const$ does not depend on $n$ (so that solutions $u_n$ belong to a fixed space $W^{1,p}(\Omega)$), structural stability results for weak solutions were obtained by Seidman [60] (see also [5]). Analogous results on entropy and renormalized solutions can be found in the works of DalMaso et al. [24], of Prignet and of Malusa [53, 47, 46]; for results in Orlicz spaces, see e.g. [13].

In the present paper, the exponent $p_n$ (and thus, the underlying function space for the solution $u_n$) varies with $n$; therefore the direct proof of convergence of weak solutions $u_n$ requires some involved assumptions on the convergence of the sequence $(f_n)_n$ of the source terms. To bypass this difficulty, we use the technique of renormalized solutions which became classical in the last decade. It turns out that the study of convergence of renormalized solutions of the problem permits to deduce convergence results for the weak solutions under much simpler assumptions on $(f_n)_n$. Basically, we only require the weak $L^1$ convergence of $f_n$ to a limit $f$, and ask that $f$ be sufficiently regular so that to allow for existence of a weak solution. Therefore the notion of renormalized solution, interesting by itself, also serves as an advanced tool for the study of weak solutions of the problem (1),(7).

For our study, the possible discrepancy between $W_0^{1,1}(\Omega) \cap W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ is a major obstacle that limits the applicability of the convergence techniques. This difficulty has been pointed out by Zhikov ([65, Lemma 3.1], see also Alkhutov, Antontsev and Zhikov [2]). Therefore full convergence results are obtained when the log-Hölder continuity of the exponent $p$ is enforced by additional assumptions (see [6]). In the general situation, we obtain partial convergence results (e.g., any of the assumptions $\forall n,\ p_n \geq p$ or $\forall n,\ p_n \leq p$ a.e. on $\Omega$ leads to a structural stability result). A related convergence result was recently obtained by Harjulehto, Hästö and Latvala in [39]; it concerns the case where $p_n \downarrow p$, in the difficult case where $p$ can attain the value 1 relevant for the image processing applications.

Let us give the outline of the paper. In § 2, we introduce some notation, state the useful properties of variable exponent Lebesgue and Sobolev spaces, and recall the properties of the Young measures associated with the weakly convergent sequences in $L^1$. In § 3 we give the definition of two kinds of solutions (in the "narrow" sense and in the "broad" sense; cf. solutions of types I and II of Zhikov [63]) and state the main results of the paper. In § 4, we prove structural stability results for weak and renormalized solutions of the $p(x)$-laplacian kind equations. This proof is the backbone of the paper. Uniqueness and generalized existence results for the $p(x)$ case are shown in § 5. In Appendix, we discuss the relevancy of the notions of broad and narrow solutions. For one particular case with a merely continuous in $x$ exponent $p$, we show that the coincidence of the two notions is, in a sense, generic with respect to the choice of $p$.

## 2. Preliminaries

Here we introduce the notation used throughout the paper, give the basic properties of variable exponent spaces and of Young measures associated with sequences weakly compact in $L^1$, and prove some auxiliary lemmas.

### *2.1. Notation*

• Throughout the paper, $\Omega$ is a bounded domain of $\mathbb{R}^N$, $N \geq 1$, with boundary $\partial\Omega$ which is assumed Lipschitz regular.
• A generic constant that only depends on $\Omega$,$b$,$p_\pm$ and on given sequences $(f_n)_n, (\mathfrak{a}_n)_n, (p_n)_n$ and $\big(\mathcal{M}_n\big)_n$ is denoted by $C$.
• For a given $r$ (which can be a constant, or a function taking values in $[p_-, p_+]$), $r'$ denotes its conjugate exponent $r/(r-1)$, $r^*$ denotes the optimal Sobolev embedding

$$r^* = \begin{cases} Nr/(N-r), & \text{if } r < N \\ \text{any real value}, & \text{if } r = N \\ +\infty, & \text{if } r > N, \end{cases} \tag{8}$$

and $(r^*)'$ denotes the conjugate exponent of $r^*$.
• For $E \subset \mathbb{R}^d$ and an $\mathbb{R}^d$-valued function $v$, the notation $\big[v \in E\big]$ will be used for the set $\big\{x \in \Omega \,\big|\, v(x) \in E\big\}$. The characteristic function of a Lebesgue measurable set $A \subset \Omega$ will be denoted by $1\!\!1_A$. The Lebesgue measure of $A$ is denoted by $\mathrm{meas}\,(A)$.
• We will extensively use the so-called truncation functions

$$T_\gamma : z \in \mathbb{R} \mapsto T_\gamma(z) = \max\{\min\{z, \gamma\}, -\gamma\}, \qquad \gamma > 0.$$

The set of $W^{2,\infty}$ functions $S : \mathbb{R} \longrightarrow \mathbb{R}$ such that $S'(\cdot)$ has a compact support will be denoted by $\mathcal{S}$; $\mathcal{S}_0$ stands for the set of all nondecreasing functions $S \in \mathcal{S}$ such that $S(0) = 0$. Notice that $T_\gamma \in \mathcal{S}_0$ for all $\gamma > 0$.
• We will also need to truncate vector-valued functions with the help of the mappings

$$h_m : \mathbb{R}^N \longrightarrow \mathbb{R}^N, \quad h_m(\lambda) = \begin{cases} \lambda, & |\lambda| \leq m \\ m\frac{\lambda}{|\lambda|}, & |\lambda| > m, \end{cases} \tag{9}$$

$m > 0$. Note the following property:

**Lemma 2.1.** *Let $h_m(\cdot)$ be defined by (9), and $\mathfrak{a}(x,z,\cdot)$ be monotone in the sense (3). Then for all $\lambda\in\mathbb{R}^N$, the map $m\mapsto \mathfrak{a}(x,z,h_m(\lambda))\cdot h_m(\lambda)$ is non-decreasing and converges to $\mathfrak{a}(x,z,\lambda)\cdot\lambda$ as $m\to+\infty$.*

Proof : The dependency of $\mathfrak{a}$ on $(x,z)$ is immaterial here, and we drop it in the notation.

Fix $\lambda\in\mathbb{R}^N$. Denote $D_m:=\mathfrak{a}(h_m(\lambda))\cdot h_m(\lambda)$. We show that for all $l>m>0$, one has $D_l-D_m\geq 0$ . The claim is evident if $|\lambda|\leq m$. For $\lambda$ such that $m<|\lambda|\leq l$,

$$D_l-D_m=\mathfrak{a}(\lambda)\cdot\lambda-\mathfrak{a}(\frac{m}{|\lambda|}\lambda)\cdot\frac{m}{|\lambda|}\lambda=(1-\frac{m}{|\lambda|})\mathfrak{a}(\lambda)\cdot\lambda+\frac{m}{|\lambda|}\left(\mathfrak{a}(\lambda)-\mathfrak{a}(\frac{m}{|\lambda|}\lambda)\right)\cdot\lambda;$$

thus we have

$$D_l-D_m\geq\frac{m}{|\lambda|}\left(1-\frac{m}{|\lambda|}\right)^{-1}\left(\mathfrak{a}(\lambda)-\mathfrak{a}(\frac{m}{|\lambda|}\lambda)\right)\cdot\left(\lambda-\frac{m}{|\lambda|}\lambda\right)\geq 0.$$

Finally, the case $|\lambda|>l$ reduces to the previous one, because $h_m(\lambda)=h_m\circ h_l(\lambda)$. $\diamond$

*2.2. Variable exponent Lebesgue and Sobolev spaces*

The solutions to the Dirichlet problem (1),(7) are sought within the variable exponent and the variable exponent Sobolev spaces $W_0^{1,\pi(\cdot)}(\Omega)$, $\dot{\mathrm{E}}^{\pi(\cdot)}(\Omega)$ defined below. For the sake of completeness, we also recall the definition of variable exponent Lebesgue spaces $L^{\pi(\cdot)}(\Omega)$.

**Definition 2.2.** *Let $\pi:\Omega\longrightarrow[1,+\infty)$ be a measurable function.*

• *$L^{\pi(\cdot)}(\Omega)$ is the space of all measurable functions $f:\Omega\longrightarrow\mathbb{R}$ such that the modular*

$$\rho_{\pi(\cdot)}(f):=\int_\Omega|f(x)|^{\pi(x)}\,dx<+\infty$$

*is finite, equipped with the so-called Luxembourg norm[2]:*

$$\|f\|_{L^{\pi(\cdot)}}:=\inf\Big\{\lambda>0\,\Big|\,\rho_{\pi(\cdot)}\big(f/\lambda\big)\leq 1\Big\}.$$

*In the sequel, we will use the same notation $L^{\pi(\cdot)}(\Omega)$ for the space $(L^{\pi(\cdot)}(\Omega))^N$ of vector-valued functions.*

• *$W^{1,\pi(\cdot)}(\Omega)$ is the space of all functions $f\in L^{\pi(\cdot)}(\Omega)$ such that the gradient $\nabla f$ of $f$ (taken in the sense of distributions) belongs to $L^{\pi(\cdot)}(\Omega)$; the space $W^{1,\pi(\cdot)}(\Omega)$ is equipped with the norm*

$$\|f\|_{W^{1,\pi(\cdot)}}:=\|f\|_{L^{\pi(\cdot)}}+\|\,\nabla f\,\|_{L^{\pi(\cdot)}}.$$

[2]one easily checks that $\|\cdot\|_{L^{\pi(\cdot)}}$ is indeed a norm on $L^{\pi(\cdot)}(\Omega)$

*Further,* $W_0^{1,\pi(\cdot)}(\Omega)$ *is the closure of* $C_0^\infty(\Omega)$ *in the norm of* $W^{1,\pi(\cdot)}(\Omega)$*, and* $\dot W^{1,\pi(\cdot)}(\Omega)$ *is the space* $W_0^{1,1}(\Omega) \cap W^{1,\pi(\cdot)}(\Omega)$ *equipped with the norm of* $W^{1,\pi(\cdot)}(\Omega)$*.*

- *In addition, we define* $\dot{\mathrm{E}}^{\pi(\cdot)}(\Omega)$ *as the set of all* $f \in W_0^{1,1}(\Omega)$ *such that* $\nabla f \in L^{\pi(\cdot)}(\Omega)$*. This space is equipped with the norm* $\|f\|_{\dot{\mathrm{E}}^{\pi(\cdot)}} := \|\nabla f\|_{L^{\pi(\cdot)}}$*.*

Notice that the definitions imply

$$W_0^{1,\infty}(\Omega) \subset W_0^{1,\pi(\cdot)}(\Omega) \subset \dot W^{1,\pi(\cdot)}(\Omega) \subset \dot{\mathrm{E}}^{\pi(\cdot)}(\Omega) \subset W^{1,1}(\Omega).$$

Moreover, $\dot W^{1,\pi(\cdot)}(\Omega)$ coincides with $\dot{\mathrm{E}}^{\pi(\cdot)}(\Omega)$ whenever $\pi(\cdot)$ is such that the Poincaré inequality

$$\|f\|_{L^{\pi}(\cdot)} \le const \, \|\nabla f\|_{L^{\pi(\cdot)}} \tag{10}$$

holds for $f \in \dot{\mathrm{E}}^{\pi(\cdot)}(\Omega)$. To the author's knowledge, no necessary and sufficient condition is known which ensures that the Poincaré inequality (10) holds even for $f \in W_0^{1,\pi(\cdot)}(\Omega)$; a sufficient condition is the continuity of $\pi$ (see Proposition 2.3 below).

The fact that, in general, $W_0^{1,\pi(\cdot)}(\Omega) \subsetneq \dot{\mathrm{E}}^{\pi(\cdot)}(\Omega)$, as well as the distinction of the associated notions of solutions to $p(x)$-laplacian kind problems (see Definition 3.1 below) go back to a series of works of Zhikov on the so-called Lavrentiev phenomenon (see in particular [63, 64, 65]). Although we have found it convenient to use notation and terminology different from those introduced by Zhikov (see also Alkhutov, Antontsev and Zhikov [2]), our work is in close correspondence with the results and ideas of the aforementioned papers.

Let us recall some useful properties of the variable exponent Lebesgue and Sobolev spaces (we follow the surveys provided by Fan and Zhao [32] and by Antontsev and Shmarev [9]).

**Proposition 2.3.** *For all measurable* $\pi : \Omega \longrightarrow [p_-, p_+]$*, the following holds.*

*(i)* $L^{\pi(\cdot)}(\Omega)$, $W^{1,\pi(\cdot)}(\Omega)$ *and* $W_0^{1,\pi(\cdot)}(\Omega)$ *are separable reflexive Banach spaces.*

*(ii)* $L^{\pi'(\cdot)}(\Omega)$ *can be identified with the dual space of* $L^{\pi(\cdot)}(\Omega)$*, and the following Hölder inequality holds:*

$$\forall f \in L^{\pi(\cdot)}(\Omega),\ g \in L^{\pi'(\cdot)}(\Omega) \quad \Big|\int_\Omega fg\Big| \le 2\|f\|_{L^{\pi(\cdot)}}\|g\|_{L^{\pi'(\cdot)}}.$$

*(iii) One has* $\rho_{\pi(\cdot)}(f) = 1$ *if and only if* $\|f\|_{L^{\pi(\cdot)}} = 1$*; further,*

$$\text{if } \rho_{\pi(\cdot)}(f) \le 1, \text{ then } \|f\|^{p_+}_{L^{\pi(\cdot)}} \le \rho_{\pi(\cdot)}(f) \le \|f\|^{p_-}_{L^{\pi(\cdot)}};$$

$$\text{if } \rho_{\pi(\cdot)}(f) \ge 1, \text{ then } \|f\|^{p_-}_{L^{\pi(\cdot)}} \le \rho_{\pi(\cdot)}(f) \le \|f\|^{p_+}_{L^{\pi(\cdot)}}.$$

*In particular, if* $(f_n)_n$ *is a sequence in* $L^{\pi(\cdot)}(\Omega)$*, then* $\|f_n\|_{L^{\pi(\cdot)}}$ *tends to zero (resp., to infinity) if and only if* $\rho_{\pi(\cdot)}(f_n)$ *tends to zero (resp., to infinity), as* $n \to \infty$*.*

*(iv)* *If, in addition, $\pi$ admits a uniformly continuous on $\Omega$ representative, then the $W_0^{1,\pi(\cdot)}$ Poincaré inequality for the norms holds:*

$$\forall f \in W_0^{1,\pi(\cdot)}(\Omega) \quad \|\nabla f\|_{L^{\pi(\cdot)}} \leq \|f\|_{L^{\pi(\cdot)}}.$$

**Proposition 2.4.** *Assume in addition that $\pi(\cdot) : \Omega \longrightarrow [p_-, p_+]$ has a representative which can be extended into a function continuous up to the boundary $\partial\Omega$ and satisfying the log-Hölder continuity assumption:*

$$\exists\, L > 0 \qquad \forall\, x, y \in \overline{\Omega},\ x \neq y, \qquad -\left(\log|x-y|\right)\left|\pi(x) - \pi(y)\right| \leq L. \tag{11}$$

*Then the following properties hold.*

*(i)* *$\mathcal{D}(\mathbb{R}^N)$ is dense in $W^{1,\pi(\cdot)}(\Omega)$, and $\mathcal{D}(\Omega)$ is dense in $\dot{W}^{1,\pi(\cdot)}(\Omega)$; in particular, the spaces $W_0^{1,\pi(\cdot)}(\Omega)$ and $\dot{W}^{1,\pi(\cdot)}(\Omega)$ coincide.*

*(ii)* *$W^{1,\pi(\cdot)}(\Omega)$ is embedded into $L^{\pi^*(\cdot)}(\Omega)$, where $\pi^*$ is the optimal Sobolev embedding exponent defined as in (8); further, if $q$ is a measurable exponent such that $ess\inf_\Omega(\pi^* - q) > 0$, then the embedding of $W^{1,\pi(\cdot)}(\Omega)$ into $L^{q(\cdot)}(\Omega)$ is compact.*

It is convenient to introduce the set of all log-Hölder continuous exponents on $\Omega$:

$$\mathcal{R}(\Omega) := \left\{ r \in C(\overline{\Omega}) \,\middle|\, r \geq 1 \text{ on } \overline{\Omega} \text{ and (11) holds} \right\}.$$

We also set $\mathcal{R}^{\pi(\cdot)}(\Omega) := \left\{ r \in \mathcal{R}(\Omega) \,\middle|\, r \leq \pi \text{ a.e. on } \Omega \right\}$. Notice that the constant exponent $p_-$ on $\Omega$ can be seen as an element of $\mathcal{R}^{\pi(\cdot)}(\Omega)$. We have the following lemma which permits us to give an equivalent definition of $\dot{\mathrm{E}}^{\pi(\cdot)}(\Omega)$.

**Lemma 2.5.** *Let $\pi : \Omega \longrightarrow [p_-, p_+]$. Let $r \in \mathcal{R}^{\pi(\cdot)}(\Omega)$. Then $\dot{\mathrm{E}}^{\pi(\cdot)}$ is continuously embedded into $W_0^{1,r(\cdot)}(\Omega)$. In particular, for all $r \in \mathcal{R}^{\pi(\cdot)}(\Omega)$ the space*

$$\left\{ f \in W_0^{1,r(\cdot)}(\Omega) \,\middle|\, \nabla f \in L^{\pi(\cdot)}(\Omega) \right\}$$

*endowed with the norm $\|f\| := \|\nabla f\|_{L^{\pi(\cdot)}} \equiv \|f\|_{\dot{\mathrm{E}}^{\pi(\cdot)}}$ coincides with $\dot{\mathrm{E}}^{\pi(\cdot)}(\Omega)$.*

Whenever $\pi \in \mathcal{R}(\Omega)$, the spaces $W_0^{1,\pi(\cdot)}(\Omega)$ and $\dot{W}^{1,\pi(\cdot)}(\Omega)$ coincide, by Proposition 2.4. We have the following stronger assertion, which permits to identify both spaces with $\dot{\mathrm{E}}^{\pi(\cdot)}(\Omega)$ (cf. [2, Theorem 2]).

**Corollary 2.6.**
*If $\pi : \overline{\Omega} \longrightarrow [p_-, p_+]$ satisfies (11), then $\dot{\mathrm{E}}^{\pi(\cdot)}(\Omega)$ and $W_0^{1,\pi(\cdot)}(\Omega)$ are Lipschitz homeomorphic. In particular, $\mathcal{D}(\Omega)$ is dense in $\dot{\mathrm{E}}^{\pi(\cdot)}(\Omega)$ whenever (11) holds.*

Indeed, in this case $\pi \in \mathcal{R}^{\pi(\cdot)}(\Omega)$, and the claim follows by Lemma 2.5.

Proof of Lemma 2.5: Take $f \in W_0^{1,1}(\Omega)$ with $\nabla f \in L^{\pi(\cdot)}(\Omega)$, and show that $f \in W_0^{1,r(\cdot)}(\Omega)$. By the choice of $r(\cdot)$ and Proposition 2.4, $W_0^{1,r(\cdot)}(\Omega)$ is metrically equivalent to $\dot{W}^{1,r(\cdot)}(\Omega)$. We have $\| \nabla f \|_{L^{r(x)}} < +\infty$, because

$$\rho_{r(\cdot)}( \nabla f) = \int_\Omega | \nabla f(x)|^{r(x)}\, dx \\ \leq \int_\Omega (1 + | \nabla f(x)|^{\pi(x)})\, dx \leq \operatorname{meas}(\Omega) + \rho_{\pi(\cdot)}( \nabla f) < +\infty.$$

Thus we only need to show that $f \in L^{r(\cdot)}(\Omega)$. Fix $\gamma \in \mathbb{R}^+$ and consider the truncated function $T_\gamma(f)$; we have $T_\gamma(f) \in L^\infty(\Omega)$ and $| \nabla T_\gamma(f)| \leq | \nabla f| \in L^{\pi(\cdot)}(\Omega) \subset L^{r(\cdot)}(\Omega)$. Thus $T_\gamma(f) \in W^{1,r(\cdot)}(\Omega)$. By assumption, $f \in W_0^{1,1}(\Omega)$; thus we also have $T_\gamma(f) \in W_0^{1,1}(\Omega)$. We conclude that $T_\gamma(f) \in W_0^{1,r(\cdot)}(\Omega)$. Thus by the choice of $r(\cdot)$ and by Proposition 2.3(iv),

$$\|T_\gamma(f)\|_{L^{r(\cdot)}} \leq const \| \nabla T_\gamma(f)\|_{L^{r(\cdot)}} \leq const \| \nabla f \|_{L^{r(\cdot)}}.$$

By the monotone convergence theorem, as $\gamma \to \infty$ we infer that $f \in L^{r(\cdot)}(\Omega)$.

We have actually shown that the identity mapping $\mathrm{Id} : \dot{\mathrm{E}}^{\pi(\cdot)} \longrightarrow W_0^{1,r(\cdot)}(\Omega)$ is a bounded operator. Thus the embedding is continuous. Finally, since $W_0^{1,r(\cdot)}(\Omega) \subset W_0^{1,1}(\Omega)$, we get $\dot{\mathrm{E}}^{\pi(\cdot)}(\Omega) \equiv \{ f \in W_0^{1,r(\cdot)}(\Omega) \,|\, \nabla f \in L^{\pi(\cdot)}(\Omega) \}$. $\diamond$

**Corollary 2.7.** *For all measurable $\pi : \Omega \longrightarrow [p_-, p_+]$, $1 < p_- \leq p_+ < +\infty$, the space $\dot{\mathrm{E}}^{\pi(\cdot)}(\Omega)$ is a separable reflexive Banach space.*

Proof : Notice that $p_- \in \mathcal{R}^{\pi(\cdot)}(\Omega)$. By Lemma 2.5, $\dot{\mathrm{E}}^{\pi(\cdot)}(\Omega)$ is continuously embedded into $W^{1,p_-}(\Omega)$ and thus in $W_0^{1,1}(\Omega)$ and in $L^{p_-}(\Omega)$. Therefore the space $\dot{\mathrm{E}}^{\pi(\cdot)}(\Omega)$ is metrically equivalent to a closed subspace of the space $S := \{ f \in L^{p_-}(\Omega) \,|\, \nabla f \in L^{\pi(\cdot)}(\Omega) \}$ supplied with the norm $\|f\|_{L^{p_-}} + \| \nabla f \|_{L^{\pi(\cdot)}}$. It follows from Proposition 2.3(i) that the space $S$ is complete, separable and reflexive. By the general results (see e.g. Brézis [18]), the claim follows. $\diamond$

In the sequel, $\left(\dot{\mathrm{E}}^{\pi(\cdot)}(\Omega)\right)^*$ denotes the dual space of $\dot{\mathrm{E}}^{\pi(\cdot)}(\Omega)$. The space $W^{-1,\pi'(x)}(\Omega)$ is the dual of $W_0^{1,\pi(x)}(\Omega)$. We use the same notation $< \cdot, \cdot >_{\pi(\cdot)}$ for the corresponding duality products.

**Corollary 2.8.** *For all $r \in \mathcal{R}^{\pi(\cdot)}(\Omega)$, the variable exponent Lebesgue space $L^{(r^*)'(\cdot)}(\Omega)$ is continuously embedded into $\left(\dot{\mathrm{E}}^{\pi(\cdot)}(\Omega)\right)^*$.*

Proof : By Lemma 2.5 and Proposition 2.4(ii) we have the embeddings

$$\dot{\mathrm{E}}^{\pi(\cdot)}(\Omega) \hookrightarrow W_0^{1,r(\cdot)}(\Omega) \hookrightarrow L^{(r^*)(\cdot)}(\Omega).$$

The result follows by duality from Proposition 2.3(ii). $\diamond$

Finally, we will need the fact that the spaces $W_0^{1,\pi(\cdot)}(\Omega)$ are stable by truncations. Notice that the analogous result for $\dot{\mathrm{E}}^{\pi(\cdot)}(\Omega)$ is evident, because $W_0^{1,1}(\Omega)$ is stable by truncations and $|\nabla T_\gamma(f)| \leq |\nabla f| \in L^{\pi(\cdot)}(\Omega)$ whenever $f \in \dot{\mathrm{E}}^{\pi(\cdot)}(\Omega)$.

**Lemma 2.9.**
*Let $f \in W_0^{1,\pi(\cdot)}(\Omega)$. Then for all $\gamma > 0$, $T_\gamma(f), T_\gamma(f)^+ \in W_0^{1,\pi(\cdot)}(\Omega)$.*

Proof : Let us treat the case of $T_\gamma$; the case of $T_\gamma^+$ is entirely similar. Notice that we can reason up to extraction of a subsequence.

Fix $\gamma > 0$ and take a sequence $(T_\gamma^m)_m$ of $C^\infty$ functions on $\mathbb{R}$, defined in such a way that $T_\gamma^m(0) = 0$, $0 \le (T_\gamma^m)' \le 1$, $(T_\gamma^m)'(\pm\gamma) = 0$, and $(T_\gamma^m)' \to T_\gamma'$ as $m \to +\infty$, uniformly on compact subsets of $\mathbb{R} \setminus \{\pm\gamma\}$. This can be done by taking $T_\gamma^m := (T_{\gamma - \frac{1}{m}}) * \delta_{\frac{1}{m}}$ with a standard sequence of mollifiers $\left(\delta_{\frac{1}{m}}\right)_m$ on $\mathbb{R}$.

Assume that $f_n \in \mathcal{D}(\Omega)$ and $\nabla f_n \to \nabla f$ in $L^{\pi(\cdot)}(\Omega)$. Set $g_n := T_\gamma^n(f_n)$; clearly, $g_n \in \mathcal{D}(\Omega)$. By Proposition 2.3(iii), we only need to show that

$$\int_\Omega \left|\nabla T_\gamma^n(f_n) - \nabla T_\gamma(f)\right|^{\pi(x)} = \rho_{\pi(\cdot)}\big(\nabla g_n - \nabla T_\gamma(f)\big) \longrightarrow 0 \quad \text{as } n \to \infty.$$

Because $\rho_{\pi(\cdot)}\big(\nabla f_n - \nabla f\big) \to 0$ as $n \to \infty$, the sequence $\left( |\nabla f_n|^{\pi(x)} \right)_n$ is equi-integrable on $\Omega$. Hence also $\left( \left|\nabla T_\gamma^n(f_n) - \nabla T_\gamma(f)\right|^{\pi(x)} \right)_n$ is equi-integrable on $\Omega$. By the Vitali theorem, it is sufficient to show that $\nabla T_\gamma^n(f_n) \longrightarrow \nabla T_\gamma(f)$ a.e. on $\Omega$ as $n \to \infty$.

The convergence of $f_n$ to $f$ in $W_0^{1,\pi(\cdot)}(\Omega)$ implies the convergence in $W_0^{1,1}(\Omega)$ and thus (for a subsequence) $f_n, \nabla f_n$ converge to $f, \nabla f$, respectively, a.e. on $\Omega$. For all $\alpha > 0$, a.e. on the set $\big[\, \big| |f| - \gamma \big| \ge \alpha \,\big]$, we have

$$\left|(T_\gamma^n)'(f_n) - (T_\gamma)'(f)\right| \le \left|(T_\gamma^n)'(f_n) - (T_\gamma)'(f_n)\right| + \left|(T_\gamma)'(f_n) - (T_\gamma)'(f)\right| \longrightarrow 0$$

as $n \to \infty$. Therefore $\nabla g_n$ converges to $\nabla T_\gamma(f)$ a.e. on $\big[\, |f| \ne \gamma \big]$.

Because for a.e. $\gamma \in \mathbb{R}$, $\operatorname{meas}\big( \big[\, |f| = \gamma \big] \big) = 0$, we conclude that $T_\gamma(f) \in W_0^{1,\pi(\cdot)}(\Omega)$ for a dense set of values of $\gamma$. Finally, notice that whenever a sequence $(\gamma_l)_l$ is such that $\gamma_l \uparrow \gamma$ as $l \to \infty$, we have $\left|\nabla T_\gamma(f) - \nabla T_{\gamma_l}(f)\right| = |\nabla f| \, 1\!\!1_{\left[\gamma_l < |f| < \gamma\right]}$. As $l \to \infty$, $\operatorname{meas}\big( \big[\gamma_l < |f| < \gamma\big] \big)$ tends to zero, so that $\rho_{\pi(\cdot)}\big( \nabla T_{\gamma_l}(f) - \nabla T_\gamma(f) \big) \longrightarrow 0$, by the Vitali Theorem. Because we can choose $(\gamma_l)_l$ such that $\big( T_{\gamma_l}(f) \big)_l \subset W_0^{1,\pi(\cdot)}(\Omega)$, we get $T_\gamma(f) \in W_0^{1,\pi(\cdot)}(\Omega)$, for all $\gamma > 0$. $\diamond$

*2.3. Young measures and nonlinear weak-* convergence*

Throughout the paper, we denote by $\delta_c$ the Dirac measure on $\mathbb{R}^d$, $d \in \mathbb{N}$, concentrated at the point $c \in \mathbb{R}^d$. By "$\Rightarrow$" we will denote the convergence in measure on $\Omega$ (of a sequence of scalar or vector-valued functions).

In the following theorem, we state the results of Ball [10], Pedregal [52] and Hungerbühler [40] which will be needed for our purposes (we limit the statement to the case of a bounded domain $\Omega$). Let us underline that the results of (ii,)(iii), expressed in terms of the in-measure convergence, are very convenient for the applications we have in mind.

**Theorem 2.10.**

*(i)* *Let $\Omega \subset \mathbb{R}^N$, $N \in \mathbb{N}$, and a sequence $(v_n)_n$ of $\mathbb{R}^d$-valued functions, $d \in \mathbb{N}$, such that $(v_n)_n$ is equi-integrable on $\Omega$. Then there exists a subsequence $(n_k)_k$ and a parametrized family $(\nu_x)_{x\in\Omega}$ of probability measures on $\mathbb{R}^d$, weakly measurable in $x$ wrt the Lebesgue measure on $\Omega$, such that for all Carathéodory function $F : \Omega \times \mathbb{R}^d \mapsto \mathbb{R}^t$, $t \in \mathbb{N}$, we have*

$$\lim_{k\to+\infty} \int_\Omega F(x, v_{n_k}(x))\, dx = \int_\Omega \int_{\mathbb{R}^d} F(x, \lambda)\, d\nu_x(\lambda) dx \tag{12}$$

*whenever the sequence $\big( F(\cdot, v_n(\cdot)) \big)_n$ is equi-integrable on $\Omega$. In particular,*

$$v(x) := \int_{\mathbb{R}^d} \lambda\, d\nu_x(\lambda)$$

*is the weak limit of the sequence $(v_{n_k})_k$ in $L^1(\Omega)$, as $k \to +\infty$.*

*The family $(\nu_x)_x$ is called the Young measure generated by the subsequence $(v_{n_k})_k$.*

*(ii)* *If $\Omega$ is of finite measure, and $(\nu_x)_x$ is the Young measure generated by a sequence $(v_n)_n$, then*

$$\nu_x = \delta_{v(x)} \text{ for a.e. } x \in \Omega \qquad \Longleftrightarrow \qquad v_n \Rightarrow v \text{ as } n \to +\infty.$$

*(iii)* *If $\Omega$ is of finite measure, $(u_n)_n$ generates a Dirac Young measure $(\delta_{u(x)})_x$ on $\mathbb{R}^{d_1}$, and $(v_n)_n$ generates a Young measure $(\nu_x)_x$ on $\mathbb{R}^{d_2}$, then the sequence $\big( (u_n, v_n) \big)_n$ generates the Young measure $\big( \delta_{u(x)} \otimes \nu_x \big)_x$ on $\mathbb{R}^{d_1+d_2}$.*

Whenever a sequence $(v_n)_n$ generates a Young measure $(\nu_x)_x$, following the terminology of [30] we will say that $(v_n)_n$ nonlinear weak-* converges, and $(\nu_x)_x$ is the nonlinear weak-* limit of the sequence $(v_n)_n$. In the case $(v_n)_n$ possesses a nonlinear weak-* convergent subsequence, we will say that it is nonlinear weak-* compact. Theorem 2.10(i) thus means that any equi-integrable sequence of measurable functions is nonlinear weak-* compact on $\Omega$.

## 3. Main definitions and results

Consider problem (1),(7) under assumptions (2)-(6).

### *3.1. Weak and renormalized solutions in the narrow and in the broad sense*

We distinguish between the following two notions of weak solutions (cf. Zhikov [63], Alkhutov, Antontsev and Zhikov [2]).

**Definition 3.1.** *Let $f \in L^1(\Omega)$.*

*(i) A function $u \in W_0^{1,p(\cdot,u(\cdot))}(\Omega)$ is called a narrow weak solution of problem (1),(7), if $b(u) \in L^1(\Omega)$ and the equation $b(u) - \operatorname{div} \mathfrak{a}(x,u,\nabla u) = f$ is fulfilled in $\mathcal{D}'(\Omega)$.*

*(ii) A function $u \in \dot{\mathrm{E}}^{p(\cdot,u(\cdot))}(\Omega)$ is called a broad weak solution of problem (1),(7), if $b(u) \in L^1(\Omega)$ and for all $\phi \in \dot{\mathrm{E}}^{p(\cdot,u(\cdot))}(\Omega) \cap L^\infty(\Omega)$,*

$$\int_\Omega b(u)\,\phi + \mathfrak{a}(x,u,\nabla u)\cdot\nabla\phi = \int_\Omega f\,\phi. \tag{13}$$

*(iii) A function $u$ like in* (ii) *which satisfies* (13) *with test functions $\phi \in W_0^{1,p(\cdot,u(\cdot))}(\Omega)$ (or, equivalently, that satisfies $b(u) - \operatorname{div} \mathfrak{a}(x,u,\nabla u) = f$ in $\mathcal{D}'(\Omega)$) is called an incomplete weak solution of problem (1),(7).*

Notice that, under the growth assumption (4), $\mathfrak{a}(x,u,\nabla u)$ belongs to $L^1(\Omega)$ and even to $L^{p'(\cdot,u(\cdot))}(\Omega)$, so the formulations (i)–(iii) make sense.

**Remark 3.2.**
(i) Narrow and broad solutions are also incomplete. Note that uniqueness of incomplete solutions cannot be expected, unless the notions of broad and narrow solutions coincide. In this paper, we are not interested in incomplete solutions.
(ii) Clearly, if $\pi(\cdot) := p(\cdot,u(\cdot))$ is such that $\mathcal{D}(\Omega)$ is dense in $\dot{\mathrm{E}}^{\pi(\cdot)}(\Omega)$, then any narrow weak solution is a broad weak solution, and vice versa. The log-Hölder continuity (11) of $\pi$ is one sufficient condition under which no distinction exists between narrow and broad solutions (see [64, 31, 57, 58]; cf. [28, 34, 26, 67] where different sufficient conditions appear). This observation is valid also for narrow and broad solutions in the renormalized sense, as introduced below.

Even for the simplest case of the Laplace equation $-\Delta u = f$, it is well known that a weak solution does not necessarily exist when $f \in L^1(\Omega)$. Since the unpublished work of Lions and Murat (see [48]; cf. [16, 24, 47, 46]), one standard way to generalize the notion of a solution (while preserving the uniqueness of a solution) has became the "renormalization procedure". Formally, it corresponds to taking in (1),(7), test functions $\phi(x)S(u(x))$ with $S \in \mathcal{S}$ (see § 2.1 for the definition of the set $\mathcal{S}$).

**Definition 3.3.** *Let $f \in L^1(\Omega)$.*

*(i) A measurable a.e. finite function $u$ on $\Omega$ is called a renormalized narrow solution of problem (1),(7), if for all $\gamma > 0$, $T_\gamma(u) \in W_0^{1,p(\cdot,u(\cdot))}(\Omega)$, and one has (for some sequence of values $M \to \infty$)*

$$\lim_{M\to\infty} \int_{\left[M<|u|<M+1\right]} \mathfrak{a}(x,u,\nabla u)\cdot\nabla u = 0, \tag{14}$$

*if, moreover, $b(u)\in L^1(\Omega)$, and for all $S\in\mathcal{S}$ one has*

$$b(u)\,S'(u) - \operatorname{div}\big(\,S'(u)\,\mathfrak{a}(x,u,\nabla u)\,\big) \\ +S''(u)\,\mathfrak{a}(x,u,\nabla u)\cdot\nabla u = f\,S'(u) \;\text{ in } \mathcal{D}'(\Omega). \tag{15}$$

*(ii) A measurable a.e. finite function $u$ on $\Omega$ is called a renormalized broad solution of problem (1),(7), if for all $\gamma>0$, $T_\gamma(u)\in\dot{\mathrm{E}}^{p(\cdot,u(\cdot))}(\Omega)$, (14) holds, $b(u)\in L^1(\Omega)$, and for all $S\in\mathcal{S}$ one has*

$$\int_\Omega b(u)\,S'(u)\,\phi + S'(u)\,\mathfrak{a}(x,u,\nabla u)\cdot\nabla\phi + S''(u)\,\mathfrak{a}(x,u,\nabla u)\cdot\nabla u\,\phi = \int_\Omega f\,S'(u)\,\phi \tag{16}$$

*for all $\phi\in\dot{\mathrm{E}}^{p(\cdot,u(\cdot))}(\Omega)\cap L^\infty(\Omega)$.*

Notice that Definition 3.3 makes sense. Indeed, let $\operatorname{supp} S'\subset[-M,M]$; then the terms $\nabla u$ in the equation

$$b(u)S'(u) - \operatorname{div}\big(\,S'(u)\mathfrak{a}(x,u,\nabla u)\,\big) + S''(u)\mathfrak{a}(x,u,\nabla u)\cdot\nabla u = fS'(u)$$

can be replaced by $\nabla T_M(u)$; hence by (4), the terms $S'(u)\mathfrak{a}(x,u,\nabla u)$ and $S''(u)\mathfrak{a}(x,u,\nabla u)\cdot\nabla u$ both lie in $L^1(\Omega)$. For the same reasons, the integral of $1\!\!1_{\left[M<|u|<M+1\right]}\,\mathfrak{a}(x,u,\nabla u)\cdot\nabla u$ is meaningful.

Standard Leray-Lions elliptic problems with $L^1$ (and even more general) source terms are well posed in the framework of renormalized solutions. The following notion of entropy solution due to Bénilan and al. [12] is an alternative way to get the well-posedness:

**Definition 3.4.** *Let $f\in L^1(\Omega)$. A measurable a.e. finite function $u$ on $\Omega$ is called an entropy narrow (respectively, broad) solution of problem (1),(7), if for all $\gamma>0$, $T_\gamma(u)\in W_0^{1,p(\cdot,u(\cdot))}(\Omega)$ (resp., $T_\gamma(u)\in\dot{\mathrm{E}}^{p(\cdot,u(\cdot))}(\Omega)$), $b(u)\in L^1(\Omega)$, and*

$$\int_\Omega b(u)\,T_\gamma(u-\phi) + \mathfrak{a}(x,u,\nabla u)\cdot\nabla T_\gamma(u-\phi) \le \int_\Omega f\,T_\gamma(u-\phi) \tag{17}$$

*for all $\phi\in\mathcal{D}(\Omega)$ (resp., for all $\phi\in\dot{\mathrm{E}}^{p(\cdot,u(\cdot))}(\Omega)\cap L^\infty(\Omega)$).*

With the techniques that became standard by now, it is not difficult to verify that entropy and renormalized solutions for the problems under consideration coincide and, moreover, (17) holds with the equality sign. For this paper, we found it convenient to work with convergence techniques proper to the renormalized solutions framework.

We have the following relations between weak and renormalized solutions.

**Proposition 3.5.**

*(i) Let $u$ be a narrow (resp., broad) weak solution of (1),(7). Then it is also a renormalized narrow (resp., broad) solution of the same problem.*

*(ii) Let $u$ be a renormalized narrow (resp., renormalized broad) solution of (1),(7). Then there exists an a.e. finite function $v : \Omega \longrightarrow \mathbb{R}^N$ such that*

$$\text{for a.e. } \gamma > 0,\ \nabla T_\gamma(u) = v 1\!\!1_{\left[|u|<\gamma\right]}. \tag{18}$$

*Moreover, $v \in L^{p(\cdot,u(\cdot))}(\Omega)$ if and only if $u$ is actually a narrow (resp., broad) weak solution of (1),(7); in this case, $u \in W_0^{1,p(\cdot,u(\cdot))}(\Omega)$ (resp., $u \in \dot{\mathrm{E}}^{p(\cdot,u(\cdot))}(\Omega)$) and $v$ is the gradient of $u$ in the sense of distributions.*

Proof of Proposition 3.5: The proof is standard. Consider e.g, the case of narrow solutions.

(i) Let $\phi \in \mathcal{D}(\Omega)$. By Lemma 2.9, we have $T_\gamma(u) \in L^\infty(\Omega) \cap W_0^{1,p(\cdot,u(\cdot))}(\Omega)$ for all $\gamma > 0$, and the definition of $\mathcal{S}$ implies that $S'$ is bounded. Hence $\psi = \phi\, S'(u)$ is an admissible test function in (1), and (15) follows. Moreover, we have $\mathfrak{a}(x,u,\nabla u)\cdot\nabla u \in L^1(\Omega)$ by (4). Since $\mathrm{meas}\left(\left[M<|u|<M+1\right]\right)$ tends to zero as $M \to \infty$, (14) follows. So a weak solution is also a renormalized one.

(ii) We can adopt e.g. the following definition of $v$:

$$\text{a.e. on } \Omega,\ v(x) := \nabla T_\gamma(u(x)), \quad \text{where } \gamma \in \mathbb{N},\ \gamma > |u(x)|.$$

This definition is consistent, because a.e. on the set $\left[\,|u| < \min\{\gamma,\hat\gamma\}\,\right]$, there holds $\nabla T_\gamma(u) = \nabla T_{\hat\gamma}(u)$. Indeed, if e.g. $\gamma < \hat\gamma$, then $T_\gamma(u) = T_\gamma(T_{\hat\gamma}(u))$, so that $\nabla T_\gamma(u) = \nabla T_{\hat\gamma}(u)\, 1\!\!1_{\left[|T_{\hat\gamma}(u)|<\gamma\right]} = \nabla T_{\hat\gamma}(u)\, 1\!\!1_{\left[|u|<\gamma\right]}$; and $\nabla T_{\hat\gamma}(u)\, 1\!\!1_{\left[|u|<\gamma\right]}$ coincides with $\nabla T_{\hat\gamma}(u)$ on $\left[\,|u| < \gamma\right]$.

Now if $v \in L^{p(\cdot,u(\cdot))}(\Omega)$, by Lemma 2.5 it follows that $(T_\gamma(u))_{\gamma>0}$ is uniformly bounded in $W_0^{1,p_-}(\Omega)$. By the standard results (see e.g. [12]), it follows that $u \in W_0^{1,p_-}(\Omega)$ and $\nabla u = v$.

Let us show that $u \in W_0^{1,p(\cdot,u(\cdot))}(\Omega)$. Because $\nabla u = v \in L^{p(\cdot,u(\cdot))}(\Omega)$, the set $\left(|\nabla T_\gamma(u)|^{p(x,u(x))}\right)_\gamma$ is equi-integrable on $\Omega$; in addition, $\nabla T_\gamma(u) \to \nabla u$ a.e. on $\Omega$ as $\gamma \to +\infty$, because $u$ is a.e. finite. Since $\left(T_\gamma(u)\right)_\gamma \subset W_0^{1,p(\cdot,u(\cdot))}(\Omega)$, by the Vitali theorem we deduce that $u \in W_0^{1,p(\cdot,u(\cdot))}(\Omega)$.

Now the weak formulation of (1) follows from (14),(15). Indeed, we take a sequence $S_M \in \mathcal{S}$ such that $\|S''_M\|_{L^\infty} \le 2$, $S'_M(z) = 1$ for $|z| < M$, and $S'(z) = 0$ for $|z| > M+1$. Then it suffices to let $M \to +\infty$; notice that the term $S''_M(u)\mathfrak{a}(x,u,\nabla u)\cdot\nabla u$ converges to zero in $L^1(\Omega)$, thanks to the constraint (14).

Thus we have shown that the $L^{p(\cdot,u(\cdot))}(\Omega)$ summability of $\nabla u$ forces a renormalized narrow solution $u$ to be a weak one. The converse statement has already been shown in (i); thus the proof of (ii) is complete. $\diamond$

**Remark 3.6.** It is clear that a broad weak solution of (1),(7) which, in addition, belongs to $W_0^{1,p(\cdot,u(\cdot))}(\Omega)$ is also a narrow weak solution of the same problem. Analogously, a renormalized broad solution of (1),(7) with truncatures $T_\gamma(u)$ in $W_0^{1,p(\cdot,u(\cdot))}(\Omega)$ is also a renormalized narrow solution of the same problem.

*3.2. The statements*

In the case of $u$-independent $p(\cdot)$, considering weak, entropy and renormalized solutions in the above narrow sense has become standard. In particular, in the case $\mathfrak{a}(x,\xi) = \nabla_\xi \Phi(x,\xi)$ for some strictly convex in $\xi$ function $\Phi : \Omega \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$, a narrow weak solution of (1),(7) is the unique minimizer of the functional

$$J : v \mapsto \int_\Omega \Big( \, B(v(x)) + \Phi(x,\nabla v(x)) - f(x)\, v(x) \, \Big)\, dx$$

in the space $W_0^{1,p(\cdot)}(\Omega)$; here $B(z) := \int_0^z b(s)\, ds$. Similarly, a broad weak solution of (1),(7) in the unique minimizer of $J$ in the space $\dot{\mathrm{E}}^{p(\cdot)}(\Omega)$. Because, in general, $W_0^{1,p(\cdot)}(\Omega)$ can be a strict closed subspace of $\dot{\mathrm{E}}^{p(\cdot)}(\Omega)$, the corresponding minimizers could be different. Notice that in the same way, one could have considered weak (variational) solutions associated e.g. with the intermediate space $\dot{W}^{1,p(\cdot)}(\Omega)$. The reason we focus on the narrow weak solutions and, in addition, introduce broad weak solutions, is the following structural stability theorem.

Roughly speaking, we prove that the class of narrow weak solutions is stable under approximation of $p(x)$ from above; and the class of broad weak solutions is stable under approximation of $p(x)$ from below[3]. As a simple illustrative example for Theorem 3.7, the reader can think of the sequence of $p_n(x)$-laplacian problems with a monotone sequence $(p_n)_n$ and a fixed source term $f_n \equiv f$, e.g. with $f \in L^{((p_-)^*)'}(\Omega)$.

**Theorem 3.7.**
*Assume $\big(\mathfrak{a}_n\big)_n$ is a sequence of diffusion flux functions of the form $\mathfrak{a}_n(x,\xi)$ such that (2),(3) hold for all $n$; assume (4),(5) hold with $C, p_\pm$ independent of $n$, and with a sequence $\big(\mathcal{M}_n\big)_n$ equi-integrable on $\Omega$. Let $p_n : \Omega \longrightarrow [p_-,p_+]$ be the associated exponents featuring in assumptions (4),(5). Assume*

$$\left|\begin{array}{l} \text{for all bounded subset } K \text{ of } \mathbb{R}^N, \\ \sup_{\xi\in K} |\mathfrak{a}_n(\cdot,\xi) - \mathfrak{a}(\cdot,\xi)| \text{ converges to zero in measure on } \Omega, \end{array}\right. \tag{19}$$

*where $\mathfrak{a}(x,\xi)$ verifies (3), and the growth and coercivity conditions (4),(5) hold with the exponent $p$ such that*

$$p_n \text{ converges to } p \text{ in measure on } \Omega. \tag{20}$$

*Finally, assume*

$$(f_n)_n \subset L^1(\Omega),\ f_n \text{ converges to } f \in L^1(\Omega) \text{ weakly in } L^1(\Omega). \tag{21}$$

*Denote by ($1_n$),(7) the problem associated with $\mathfrak{a}_n, f_n$. The following statements hold.*

[3] in Proposition 5.1, we further argument in favor of relevancy of the notions of broad and narrow solutions

(i) *Assume $p_n \le p$ a.e. on $\Omega$. Assume $(u_n)_n$ is a sequence of broad weak solutions of the associated problems ($1_n$),(7). Whenever $f \in \left(\dot{\mathrm{E}}^{p(\cdot)}(\Omega)\right)^*$, there exists $u \in \dot{\mathrm{E}}^{p(\cdot)}(\Omega)$ such that $u_n$,$\nabla u_n$ converge to $u$,$\nabla u$, respectively, a.e. on $\Omega$, as $n \to \infty$. The function $u$ is a broad weak solution of the problem (1),(7) associated with the diffusion flux $\mathfrak{a}$ and the source term $f$.*

(ii) *Assume $p_n \ge p$ a.e. on $\Omega$. Assume $(u_n)_n$ is a sequence of narrow weak solutions of the associated problems ($1_n$),(7). Whenever $f \in W^{-1,p'(\cdot)}(\Omega)$, there exists $u \in W_0^{1,p(\cdot)}(\Omega)$ such that $u_n$,$\nabla u_n$ converge to $u$,$\nabla u$, respectively, a.e. on $\Omega$; moreover, for all $\gamma > 0$, $T_\gamma(u_n)$ converge to $T_\gamma(u)$ strongly in $W_0^{1,p(\cdot)}(\Omega)$, as $n \to \infty$[4]. The function $u$ is a narrow weak solution of the problem (1),(7) associated with the diffusion flux $\mathfrak{a}$ and the source term $f$.*

For general convergent in measure sequences $(p_n)_n$, we can only prove a continuous dependence result for broad weak solutions, under the following technical hypothesis:

$$\left|\begin{array}{l}\text{the space } \displaystyle\bigcup_{N\in\mathbb{N}} \bigcap_{n=N}^{\infty} \dot{\mathrm{E}}^{p_n(\cdot)}(\Omega) \text{ contains a subset } \mathcal{E}(\Omega) \text{ weakly dense in } \dot{\mathrm{E}}^{p(\cdot)}(\Omega);\\ \text{moreover, } \mathcal{E}(\Omega) \subset L^\infty(\Omega) \text{ and for all } e \in \mathcal{E}(\Omega), \text{ the equi-integrability property}\\ \text{holds: } \displaystyle\lim_{\mathrm{meas}\,(E)\to 0} \sup_{n\in\mathbb{N}} \int_E |\nabla e(x)|^{p_n(x)}\,dx = 0.\end{array}\right. \tag{22}$$

**Theorem 3.8.** *Under the assumptions of Theorem 3.7 ( those preceding statements (i),(ii) ), let $(u_n)_n$ be a sequence of broad weak solutions of the problems ($1_n$),(7) associated with $\mathfrak{a}_n, f_n$ and the exponents $p_n$. Recall that $\mathfrak{a}, p, f$ are the limits of $\mathfrak{a}_n, p_n, f_n$ in the sense (19)-(21).*

*Assume the exponents $p, (p_n)_n$ satisfy (22).*

*Whenever $f \in \left(\dot{\mathrm{E}}^{p(\cdot)}(\Omega)\right)^*$, there exists $u \in \dot{\mathrm{E}}^{p(\cdot)}(\Omega)$ such that $u_n$,$\nabla u_n$ converge to $u$,$\nabla u$, respectively, a.e. on $\Omega$, as $n \to \infty$. The function $u$ is a broad weak solution of the problem (1),(7) associated with the diffusion flux $\mathfrak{a}$ and the source term $f$.*

**Remark 3.9.** In the case $\mathcal{D}(\Omega)$ is dense in $\dot{\mathrm{E}}^{p(\cdot)}(\Omega)$, (22) holds with $\mathcal{E}(\Omega) = \mathcal{D}(\Omega)$. A particular case is that of a constant $p$. More generally, by Corollary 2.6, it suffices that $p(\cdot)$ satisfy the log-Hölder continuity condition (11) (see [64, 31, 57, 58, 32]). Other sufficient conditions are given in the litterature (see in particular Edmunds and Rákosník [28], Fan, Wang and Zhao [34], Diening,

[4]in the case (ii), a stronger assumption on the convergence of $(f_n)_n$ leads to the strong convergence of $u_n$ to $u$ in $W_0^{1,p(\cdot)}(\Omega)$ (which can be seen as optimal wrt the *a priori* regularity of $u$) : see Remark 4.1 in § 4.

Hästö and Nekvinda [26], Zhikov [67]). If the space dimension $N$ is one, no condition is needed.

The second situation where (22) is trivially satisfied is the case where $p_n(\cdot) \le p(\cdot)$ a.e. on $\Omega$; indeed, it suffices to take $\mathcal{E}(\Omega) = \dot{\mathrm{E}}^{p(\cdot)}(\Omega)$. This is precisely the case of Theorem 3.7(i).

Let us stress that although Theorems 3.7,3.8 assert on convergence of weak (broad or narrow) solutions, in their proof the device of renormalized solutions is used. This is done in order to achieve the simplest assumptions on the convergence of $(f_n)_n$. Namely, we only require the weak $L^1$ convergence of $f_n$ and put a condition on their limit $f$ which ensures that a weak solution makes sense. As a matter of fact, at the same cost as Theorem 3.8, we obtain the following generalization, which is optimal for the $L^1$ framework chosen in this paper.

**Theorem 3.10.**

*(i)* *Take the assumptions preceding statements (i),(ii) of Theorem 3.7. Let $(u_n)_n$ be a sequence of renormalized broad solutions of the problems $(1_n)$,(7) associated with $\mathfrak{a}_n, f_n$ and the exponents $p_n$. Recall that $\mathfrak{a}, p, f$ are the limits of $\mathfrak{a}_n, p_n, f_n$ in the sense (19)-(21).*

*Assume the exponents $p, (p_n)_n$ satisfy (22).*

*Then there exists a measurable function $u$ on $\Omega$ such that $u_n$,$\nabla u_n$ converge to $u$,$\nabla u$, respectively, a.e. on $\Omega$, as $n \to \infty$. The function $u$ is a renormalized broad solution of the problem (1),(7) associated with the diffusion flux $\mathfrak{a}$ and the source term $f$.*

*(ii)* *In the above assumptions, replace the assumption that $u_n$ are renormalized broad solutions by the assumption that $u_n$ are renormalized narrow solutions of problems $(1_n)$,(7) associated with $\mathfrak{a}_n, f_n$ and the exponents $p_n$.*

*Replace assumption (22) by the assumption that $p_n \ge p$ a.e. on $\Omega$.*

*Then there exists a measurable function $u$ on $\Omega$ such that $u_n$,$\nabla u_n$ converge to $u$,$\nabla u$, respectively, a.e. on $\Omega$; moreover, for all $\gamma > 0$, $T_\gamma(u_n)$ converge to $T_\gamma(u)$ strongly in $W_0^{1,p(\cdot)}(\Omega)$, as $n \to \infty$. The function $u$ is a renormalized narrow solution of the problem (1),(7) associated with the diffusion flux $\mathfrak{a}$ and the source term $f$.*

Now let us point out that for all source terms in $L^1(\Omega)$, renormalized broad solutions and narrow solutions of the problems considered in Theorem 3.10 do exist. The situation with weak solutions is different: unless $p_- > N$, their existence requires additional restrictions of $f$. Notice that in Theorems 3.7,3.8, we do not assert the existence of a (narrow or broad) weak solution $u_n$ to $(1_n)$,(7), but assume it. The existence result below is natural with respect to the standard variational setting; now we allow for an explicit dependency of $\mathfrak{a}$ on $u$, provided the associated exponent $p$ remains independent of $u$[5].

[5] it is not difficult to generalize the proof of the above continuity theorems also to this case;

**Theorem 3.11.**
*Assume* $\mathfrak{a} = \mathfrak{a}(x, \xi)$ *satisfy* (2),(3) *with* $p : \Omega \longrightarrow [p_-, p_+]$ *measurable,* $1 < p_- \leq p_+ < +\infty$*. Assume the following* $p(x)$*-growth assumption:*

$$|\mathfrak{a}(x, z, \xi)|^{p'(x)} \leq C \left( |\xi|^{p(x)} + \mathcal{M}(x) + \mathcal{L}\big(zb(z) + |z|^{r^*(\cdot)}\big) \right), \tag{23}$$

*and the coercivity assumption* (5) *can also be relaxed to*

$$\mathfrak{a}(x, z, \xi) \cdot \xi \geq \frac{1}{C} |\xi|^{p(x)} - \mathcal{M}(x) - \mathcal{L}\big(zb(z) + |z|^{r^*(\cdot)}\big). \tag{24}$$

*Here* $\mathcal{M} \in L^1(\Omega)$*, the exponent* $r(\cdot)$ *belongs to* $\mathcal{R}^{p(\cdot)}(\Omega)$*, and* $\mathcal{L} : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *is a sublinear function in the sense that* $\lim_{t\to\infty} \mathcal{L}(t)/t = 0$*.*

*(i) Assume* $f \in L^1(\Omega)$*. Then there exists a measurable function* $u$ *on* $\Omega$ *such that* $u$ *is a renormalized broad solution of* (1),(7)*.*

*The same claim is true for the existence of a renormalized narrow solution.*

*(ii) Assume* $f \in L^1(\Omega) \cap W_0^{-1,p'(\cdot)}(\Omega)$*.*
*Then there exists a narrow weak solution of* (1),(7)*.*

*(iii) Assume* $f \in L^1(\Omega) \cap \big(\dot{\mathrm{E}}^{p(\cdot)}(\Omega)\big)^*$*.*
*Then there exists a broad weak solution of* (1),(7)*.*

We infer the existence results from the above structural stability theorems (or rather, we slightly adapt their proofs).

Uniqueness of a weak (resp., renormalized) broad solution for the case of $u$-independent diffusion flux function $\mathfrak{a}$ can be shown in exactly the same way as the uniqueness of a corresponding narrow solution (we refer to [59, 11, 51] for the uniqueness results on renormalized and entropy narrow solutions). For the sake of completeness, let us state the corresponding result.

**Theorem 3.12.** *Assume* $\mathfrak{a} = \mathfrak{a}(x, \xi)$ *satisfy* (2)-(5) *with* $p : \Omega \longrightarrow [p_-, p_+]$ *measurable,* $1 < p_- \leq p_+ < +\infty$*. Let* $f \in L^1(\Omega)$*. Consider any of the notions (narrow or broad; weak or renormalized) of solution to problem* (1),(7)*. There exists at most one solution of* (1),(7)*.*

*Moreover, if* $u_f, u_{\hat{f}}$ *are solutions, in the same sense, corresponding to the data* $f, \hat{f}$*, then the following* $L^1$ *contraction and comparison principle holds:*

$$\int_\Omega \big( b(u_f) - b(u_{\hat{f}}) \big)^+ \leq \int_\Omega (f - \hat{f}) \operatorname{sign}^+(u_f - u_{\hat{f}}) \leq \int_\Omega (f - \hat{f})^+. \tag{25}$$

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but, as it is shown in the proof of Theorem 3.11, instead of doing this we can simply consider the terms $\mathfrak{a}_n(x, u_n(x), \nabla u_n)$ as being of the form $\tilde{\mathfrak{a}}_n(x, \nabla u_n)$, and apply Theorems 3.7,3.8 as they are stated above.

Notice that analogous uniqueness result can be shown also in the case $\mathfrak{a}$ depends on $u$, but $p$ remains independent of $u$ (see the existence Theorem 3.11 above); but one needs a Lipschitz or Hölder continuity assumption on $\mathfrak{a}(x,\cdot,\xi)$ in the spirit of [3, 50]. Let us stress that for $p\equiv const$, more general uniqueness results are available (see in particular [20]); they are based on the Kruzhkov and Carrillo doubling of variables technique. To the best of the authors knowledge, adaptation of this technique to the case of a variable exponent $p(x)$ remains an open problem.

## 4. Continuous dependence on the variable exponent $p_n(x)$

We prove Theorem 3.8 and then indicate the additional arguments needed for Theorem 3.7 and Theorem 3.10. Before starting, let us precise the role of the truncations and of the renormalized formulation (16) in the below proof. Truncations are used in order to allow for a passage to the limit in the term $\displaystyle\int_\Omega f_n\, T_\gamma(u_n)$; this is a part of the monotonicity-based identification argument. When the identification is completed, we actually show that $u$ is a renormalized broad solution of the limit problem. Then the assumption that $f\in\big(\dot{\mathrm{E}}^{p(\cdot)}(\Omega)\big)^*$ permits to assert that the limit $u$ turns out to be a broad weak solution. If we only used the weak formulation (13), the corresponding term would be $\displaystyle\int_\Omega f_n\, u_n$; the passage to the limit in this term would require quite involved assumptions on the sequence $(f_n)_n$.

Proof of Theorem 3.8:

The proof is split into several steps. In Claims 1,2 we gather the uniform in $n$ estimates on the truncated solutions $T_\gamma(u_n)$. Claims 3—8 are technical; they contain a kind of compactness result which is expressed in terms of the Young measures corresponding to the truncation sequences $(T_\gamma(u_n))_n$. Claim 9 is the heart of the proof and its most delicate point; here assumption (22) is needed, and the distinction between narrow and broad solutions becomes crucial. Claims 10—12 contain the reduction argument for the Young measures and its consequences, including the strong convergence of $\nabla u_n$. In Claims 13—15, it is shown that $u$ is a renormalized, and then a weak, solution to problem (1),(7).

Throughout the proof, we reason up to an extracted subsequence of $(u_n)_n$.

- Claim 1 : Let $\gamma>0$. Then the sequence $\|T_\gamma(u_n)\|_{\dot{\mathrm{E}}^{p_n(\cdot)}}$ is bounded.

Because $u_n\in W_0^{1,1}(\Omega)$ and $T_\gamma$ is Lipschitz continuous, we have $\nabla T_\gamma(u_n)=\nabla u_n 1\!\!1_{\left[|u_n|\le\gamma\right]}$. Let us show that $T_\gamma(u_n)\in\dot{\mathrm{E}}^{p_n(\cdot)}(\Omega)$ and there exists $C$, independent of $n$ and $\gamma$, such that

$$\int_\Omega |\,\nabla T_\gamma(u_n)|^{p_n(x)}=\int_{\left[|u_n|\le\gamma\right]} |\,\nabla u_n|^{p_n(x)}\,dx\le C\,\gamma. \tag{26}$$

It is clear that $T_\gamma(u_n) \in W_0^{1,1}(\Omega)$, and $|\nabla T_\gamma(u_n)| \leq |\nabla u_n| \in L^{p_n(\cdot)}(\Omega)$. Thus, taking $T_\gamma(u_n)$ for the test function in the broad weak formulation of ($1_n$),(7), by assumption (5) and the monotonicity of $b$ we infer

$$\int_{\big[|u_n|\leq\gamma\big]} |\,\nabla u_n|^{p_n(x)} \leq C\gamma \|f_n\|_{L^1(\Omega)}. \tag{27}$$

Since $(f_n)_n$ is weakly convergent in $L^1(\Omega)$, the right-hand side of (27) is bounded by $C\,\gamma$. By Proposition 2.3(iii), this yields $\|T_\gamma(u_n)\|_{\dot{\mathrm{E}}^{p_n(\cdot)}} \leq C\ \max\{\gamma^{1/p_-}, \gamma^{1/p_+}\}$. Hence the claim follows.

• Claim 2 : the sequence $(u_n)_n$ satisfies the estimate

$$\lim_{\gamma\to\infty} \sup_n \int_{\big[\gamma<|u_n|<\gamma+1\big]} |\,\nabla u_n|^{p_n(x)} = 0. \tag{28}$$

For the proof, we replace in the argument of Claim 1, the test function $T_\gamma(u_n)$ by the test function $T_{\gamma+1}(u_n) - T_\gamma(u_n)$. Because it is supported on $\big[|u_n| \geq \gamma\big]$ and its $L^\infty$ norm is bounded by one, we infer

$$\int_{\big[\gamma<|u_n|<\gamma+1\big]} |\,\nabla u_n|^{p_n(x)} \ \leq \int_{\big[|u_n|\geq\gamma\big]} |f_n|\,.$$

Being weakly convergent in $L^1(\Omega)$, the sequence $(f_n)_n$ is also equi-integrable on $\Omega$; therefore, (28) will follow if we show that meas$\big(\,\big[|u_n| \geq \gamma\big]\,\big)$ tends to zero as $\gamma \to +\infty$ uniformly in $n$. Now by Claim 1 and the Poincaré inequality applied in $W_0^{1,p_-}(\Omega)$, we have

$$\begin{aligned}\text{meas}\big(\,\big[|u_n| \geq \gamma\big]\,\big) &\leq \frac{1}{\gamma^{p_-}} \int_\Omega |T_\gamma(u_n)|^{p_-} \leq \frac{C}{\gamma^{p_-}} \int_\Omega |\,\nabla T_\gamma(u_n)|^{p_-} \\ &\leq \frac{C}{\gamma^{p_-}} \int_\Omega \big(\,1 + |\,\nabla T_\gamma(u_n)|^{p_n(x)}\,\big) \leq C\frac{1+\gamma}{\gamma^{p_-}}.\end{aligned} \tag{29}$$

Thus $\lim_{\gamma\to\infty} \sup_n \text{meas}\big(\,\big[|u_n| \geq \gamma\big]\,\big) = 0$, which proves (28).

• Claim 3 : there exists a measurable, a.e. finite function $u$ on $\Omega$ such that for all $\gamma \in \mathbb{N}$, $T_\gamma(u) \in W_0^{1,p_-}(\Omega)$ and the sequence $(u_n)_n$ admits a subsequence satisfying, for all $\gamma \in \mathbb{N}$, $T_\gamma(u_n) \rightharpoonup T_\gamma(u)$ in $W_0^{1,p_-}(\Omega)$. Furthermore, $u_n \to u$ a.e. on $\Omega$, and $\nabla T_\gamma(u_n)$ converges to a Young measure $\nu_x^\gamma(\lambda)$ on $\mathbb{R}^N$ in the sense of the nonlinear weak-* convergence, and

$$\nabla T_\gamma(u) = \int_{\mathbb{R}^N} \lambda\, d\nu_x^\gamma(\lambda). \tag{30}$$

Indeed, the bound obtained in Step 1 implies that

$$\|T_\gamma(u_n)\|^{p_-}_{W_0^{1,p_-}} = \int_\Omega |\,\nabla T_\gamma(u_n)|^{p_-} \leq \int_\Omega (1 + |\,\nabla T_\gamma(u_n)|^{p_n(x)}) \leq C(\gamma).$$

Extract a (not relabelled) subsequence such that for all $\gamma \in \mathbb{N}$, $T_\gamma(u_n) \rightharpoonup z_\gamma$ in $W_0^{1,p_-}(\Omega)$ and $T_\gamma(u_n) \to z_\gamma$ a.e. on $\Omega$. Then we can define

$$\text{a.e. on } \Omega,\ u(x) := \lim_{\gamma\to\infty} z_\gamma(x). \tag{31}$$

The function $u$ is well defined, because for $\gamma, \hat\gamma \in \mathbb{N}$ such that $\gamma < \hat\gamma$, $T_\gamma(u_n) \equiv T_\gamma(T_{\hat\gamma}(u_n))$ converges a.e. on $\Omega$ to $z_\gamma$ and to $T_\gamma(z_{\hat\gamma})$. By the uniqueness of the limit, $z_\gamma \equiv T_\gamma(z_{\hat\gamma})$; one easily deduces that for a.e. $x \in \Omega$, the sequence $(z_\gamma(x))_\gamma$ is monotone and thus converges to a limit in $\overline{\mathbb{R}}$. Finally, assume that $\operatorname{meas}\big(\,\big[|u| = \infty\big]\,\big) = \alpha > 0$. Then for all $\gamma \in \mathbb{N}$, $\operatorname{meas}\big(\,\big[|z_\gamma| = \gamma\big]\,\big) \geq \alpha$. Notice that for all $n$, $\big[|u_n| \geq \gamma - 1\big] \supset \big[|z_\gamma| \geq \gamma - 1/2\big] \cap \big[|T_\gamma(u_n) - z_\gamma| \leq 1/2\big]$, so that

$$\operatorname{meas}\big(\,\big[|u_n| \geq \gamma-1\big]\,\big) \geq \operatorname{meas}\big(\,\big[|z_\gamma| \geq \gamma-1/2\big]\,\big) - \operatorname{meas}\big(\,\big[|T_\gamma(u_n)-z_\gamma| > 1/2\big]\,\big).$$

As $n \to \infty$, we infer $\operatorname{meas}\big(\,\big[|u_n| \geq \gamma - 1\big]\,\big) \geq \operatorname{meas}\big(\,\big[|z_\gamma| = \gamma\big]\,\big) \geq \alpha > 0$. As $\gamma \to +\infty$, we get a contradiction with estimate (29). This proves that $u$ is a.e. finite on $\Omega$.

Notice that the a.e. convergence of $u_n$ to $u$ follows from the a.e. convergence of $T_\gamma(u_n)$ to $T_\gamma(u)$ for all $\gamma \in \mathbb{N}$. Further, formula (31) means that for all $\gamma \in \mathbb{N}$, $T_\gamma(u) = z_\gamma \in W^{1,p_-}(\Omega)$. In particular, $\nabla T_\gamma(u_n)$ weakly converges in $L^{p_-}(\Omega)$ to the function $\nabla T_\gamma(u)$. Extracting if necessary a further subsequence, by Theorem 2.10(i), we infer the existence of a nonlinear weak-* limit $\nu_x(\lambda)$ of $\big(\,\nabla T_\gamma(u_n)\,\big)_n$ and the representation formula (30).

• Claim 4 : for all $\gamma \in \mathbb{N}$, $|\lambda|^{p(x)}$ is integrable with respect to the measure $d\nu_x^\gamma(\lambda)\,dx$ on $\mathbb{R}^N \times \Omega$; moreover, $T_\gamma(u) \in \dot{\mathrm{E}}^{p(x)}(\Omega)$.

By assumption (20) and Theorem 2.10(ii),(iii), for all $\gamma \in \mathbb{N}$ the sequence $(p_n,\, \nabla T_\gamma(u_n))_n$ converges to the Young measure $\mu_x^\gamma$ on $\mathbb{R}\times\mathbb{R}^N$ equal to $\delta_{p(x)}\otimes\nu_x^\gamma$.

Then we apply the nonlinear weak-* convergence property (12) to the function

$$F\ :\ \big(\,x, (\lambda_0, \lambda)\,\big) \in \Omega \times (\mathbb{R} \times \mathbb{R}^N)\ \mapsto\ |h_m(\lambda)|^{\lambda_0},$$

where $(h_m)_m$ is the sequence of truncations defined by (9). Hence

$$\begin{aligned}\int_{\Omega\times\mathbb{R}^N} |h_m(\lambda)|^{p(x)}\ d\nu_x^\gamma(\lambda)\,dx &= \int_{\Omega\times(\mathbb{R}\times\mathbb{R}^N)} |h_m(\lambda)|^{\lambda_0}\ d\mu_x^\gamma(\lambda_0, \lambda)\,dx\\ = \lim_{n\to\infty}\int_\Omega |h_m(\nabla T_\gamma(u_n))|^{p_n(x)}\,dx &\leq \liminf_{n\to\infty}\int_\Omega |\nabla T_\gamma(u_n)|^{p_n(x)}\,dx \leq C(\gamma).\end{aligned}$$

As $m$ tends to $+\infty$, by the monotone convergence theorem we infer that $|\lambda|^{p(x)}$ is integrable on $\mathbb{R}^N \times \Omega$ wrt the measure $d\nu_x(\lambda)\,dx$. Hence we also deduce that $T_\gamma(u) \in \dot{\mathrm{E}}^{p(x)}(\Omega)$. Indeed, $T_\gamma(u) \in W_0^{1,p_-}(\Omega)$, and, in addition,

$$\int_\Omega |\,\nabla T_\gamma(u)|^{p(x)} = \int_\Omega \Big|\int_{\mathbb{R}^N} \lambda\, d\nu_x^\gamma(\lambda)\Big|^{p(x)}\ dx \leq \int_{\Omega\times\mathbb{R}^N} |\lambda|^{p(x)}\ d\nu_x^\gamma(\lambda)\,dx < +\infty$$

thanks to the representation formula (30) and to the Jensen inequality.

• Claim 5 : we have (for a sequence $M \to +\infty$)

$$\lim_{M\to+\infty} \int_\Omega \Big| \nabla(T_{M+1}(u) - T_M(u)) \Big|^{p(x)} = 0. \tag{32}$$

The proof uses the same ideas as in Claims 3,4 above. We extract a further subsequence of $(u_n)_n$ such that for all $M$, $T_{M+1}(u_n) - T_M(u_n)$ converges to $T_{M+1}(u) - T_M(u)$ a.e. on $\Omega$ and weakly in $W_0^{1,p_-}(\Omega)$. Introducing the Young measure corresponding to $\nabla\big(T_{M+1}(u) - T_M(u)\big)$, with the technique of Claim 4 we deduce that $\nabla\big(T_{M+1}(u) - T_M(u)\big) \in L^{p(\cdot)}(\Omega)$ and its modular is upper bounded by

$$\sup_n \int_\Omega |\nabla(T_{M+1}(u_n) - T_M(u_n))|^{p_n(x)}\,dx = \sup_n \int_{\big[M<|u_n|<M+1\big]} |\nabla u_n|^{p_n(x)}\,dx.$$

Using estimate (28), we deduce (32).

• Claim 6 : for all $\gamma \in \mathbb{N}$, the sequence $(\chi_n^\gamma)_n$, $\chi_n^\gamma(x) := \mathfrak{a}_n(x,\nabla T_\gamma(u_n(x)))$, is relatively weakly compact in $L^1(\Omega)$.

Indeed, it suffices to show that $(\chi_n^\gamma)_n$ is equi-integrable on $\Omega$. By assumption (4) and Proposition 2.3(ii), we get for all measurable $E \subset \Omega$,

$$\begin{aligned}\int_E |\mathfrak{a}_n(x,\nabla T_\gamma(u_n))| &\le C \int_E \big(1 + \mathcal{M}(x) + |\nabla T_\gamma(u_n)|^{p_n(x)-1}\big) \\ &\le C \int_E (1 + \mathcal{M}(x)) \;+\; C \|\,|\nabla T_\gamma(u_n)|^{p_n(x)-1}\|_{L^{p'_n(x)}} \|1\!\!1_E\|_{L^{p_n(x)}}.\end{aligned}$$

The first term in the right-hand side above is small for meas $(E)$ small. Further, by Proposition 2.3(iii), the norm $\|1\!\!1_E\|_{L^{p_n(\cdot)}}$ does not exceed the value

$$\max\big\{(\rho_{p_n(\cdot)}(1\!\!1_E))^{\frac{1}{p_-}}, (\rho_{p_n(\cdot)}(1\!\!1_E))^{\frac{1}{p_+}}\big\} = \max\big\{\text{meas}\,(E)^{\frac{1}{p_-}}, \text{meas}\,(E)^{\frac{1}{p_+}}\big\}.$$

Similarly,

$$\begin{aligned}\|\,|\nabla T_\gamma(u_n)|^{p_n(x)-1}\|_{L^{p'_n(x)}} &\le \max\Big\{1, \Big(\rho_{p'_n(\cdot)}\big(|\nabla T_\gamma(u_n)|^{p_n(x)-1}\big)\Big)^{1/p_-}\Big\} \\ &= \max\Big\{1\,, \Big(\int_\Omega |\nabla T_\gamma(u_n)|^{p_n(x)}\Big)^{1/p_-}\Big\}.\end{aligned}$$

Now the claim follows from estimate (26).

• Claim 7 : the weak $L^1$ limit $\chi^\gamma$ of (a subsequence of) $(\chi_n^\gamma)_n$ belongs to $L^{p'(\cdot)}(\Omega)$ , and one has for a.e $x \in \Omega$,

$$\chi^\gamma(x) = \int_{\mathbb{R}^N} \mathfrak{a}(x,\lambda)\,d\nu_x^\gamma(\lambda). \tag{33}$$

For the proof, set $v_n := T_\gamma(u_n)$; recall that $\chi_n^\gamma(x) = \mathfrak{a}_n(x,\nabla v_n(x))$. Consider auxiliary functions $\bar{\chi}_n^\gamma(x) := \mathfrak{a}(x,\nabla v_n(x) 1\!\!1_{R_n}(x))$, where we have introduced the set $R_n := \big\{\, x \in \Omega \,\big|\, |p(x) - p_n(x)| < 1/2 \,\big\}$.

Let us show that the sequence $(\bar\chi_n^\gamma)_n$ is equi-integrable on $\Omega$. By (4), we have

$$\int_E |\bar\chi_n^\gamma| \le C\int_E \big(1+\mathcal{M}_n(x)\big) \;+\; \int_{E\cap R_n} |\,\nabla v_n(x)|^{p(x)-1}.$$

The first term in the right-hand side is uniformly small for $\mathrm{meas}\,(E)$ small. Next, because for $x\in R_n$, $p(x)\le p_n(x)+1/2$, we have

$$\begin{aligned}\int_{E\cap R_n} |\,\nabla v_n(x)|^{p(x)-1} &\le \int_E (1+|\,\nabla v_n(x)|^{p_n(x)-1/2})\\ &\le \mathrm{meas}\,(E) + C\|\,\nabla v_n\|_{L^{p_n(\cdot)}}\|1\!\!1_E\|_{L^{2p_n(\cdot)}}\end{aligned} \tag{34}$$

by Proposition 2.3(ii). Like in the preceding Step, we have

$$\begin{aligned}\|1\!\!1_E\|_{L^{2p_n(\cdot)}} &\le \max\big\{(\rho_{2p_n(\cdot)}(1\!\!1_E))^{\frac{1}{2p_-}}, (\rho_{2p_n(\cdot)}(1\!\!1_E))^{\frac{1}{2p_+}}\big\}\\ &= \max\big\{\mathrm{meas}\,(E)^{\frac{1}{2p_-}}, \mathrm{meas}\,(E)^{\frac{1}{2p_+}}\big\}.\end{aligned}$$

The right-hand side of (34) is uniformly small for $\mathrm{meas}\,(E)$ small, and the equi-integrability of $(\bar\chi_n^\gamma)_n$ follows. Thus (for a subsequence) $\bar\chi_n^\gamma$ converges weakly to some $\bar\chi^\gamma$ in $L^1(\Omega)$ as $n\to+\infty$.

Now we show that $\bar\chi^\gamma=\chi^\gamma$. This follows from the fact that $\bar\chi_n^\gamma-\chi_n^\gamma$ tends strongly to zero in $L^1(\Omega)$, which we now prove.

Indeed, fix $\alpha>0$. Due to the uniform boundedness of $\int_\Omega |\,\nabla v_n(x)|^{p_n(x)}$ and hence of $\int_\Omega |\,\nabla v_n(x)|$, it follows by the Chebyshev inequality that the measure of the set $\sup_n \mathrm{meas}\,\big(\big[|\nabla v_n|>L\big]\big)$ tends to zero as $L\to\infty$. Therefore, due to the equi-integrability of both $(\chi_n^\gamma)_n$ and $(\bar\chi_n^\gamma)_n$, there exists $L=L(\alpha)$ such that for all $n$, $\int_{\big[|\nabla v_n|>L\big]} |\bar\chi_n^\gamma-\chi_n^\gamma|<\alpha/4$.

Thanks to assumption (19) and by the aforementioned equi-integrability argument, for all $\sigma>0$ there exists $n_0=n_0(\sigma,L)\in\mathbb{N}$ such that for all $n>n_0$,

$$\int_{\big[x\in\Omega\,\mid\,\sup_{|\lambda|\le L}|\mathfrak{a}_n(x,\lambda)-\mathfrak{a}(x,\lambda)|\ge\sigma\big]} |\bar\chi_n^\gamma-\chi_n^\gamma|<\alpha/4.$$

By the definition of $\chi_n^\gamma,\bar\chi_n^\gamma$, we have $\bar\chi_n^\gamma-\chi_n^\gamma=\mathfrak{a}(x,\nabla v_n)-\mathfrak{a}_n(x,\nabla v_n)$ on the set $R_n$. We now reason on the set

$$R_n^{L,\sigma}:=\Big\{\,x\in R_n\,\Big|\,\sup_{|\lambda|\le L}|\mathfrak{a}(x,\lambda)-\mathfrak{a}_n(x,\lambda)|<\sigma,\;|\,\nabla v_n|\le L\,\Big\}.$$

There holds

$$\int_{R_n^{L,\sigma}} |\bar\chi_n^\gamma-\chi_n^\gamma| \le \int_{R_n^{L,\sigma}} \sup_{|\lambda|\le L}|\mathfrak{a}_n(x,\lambda)-\mathfrak{a}(x,\lambda)| \;\le\; \sigma\,\mathrm{meas}\,(\Omega).$$

Choosing $\sigma = \sigma(\alpha) < \alpha/(4\text{meas}\,(\Omega))$, we obtain $\int_{R_n} |\bar{\chi}_n^\gamma - \chi_n^\gamma| \leq 3\alpha/4$ for all $n > n_0(\sigma(\alpha), L(\alpha))$. To conclude, note that $\int_{\Omega\setminus R_n} |\bar{\chi}_n^\gamma - \chi_n^\gamma| = \int_{\Omega\setminus R_n} |\chi_n^\gamma| \leq \alpha/4$ for sufficiently large $n$, because meas $(\Omega \setminus R_n)$ tends to zero as $n \to \infty$ by the assumptions of the theorem.

Let us show the representation formula for $\chi^\gamma$. To this end, notice that $\nabla v_n(x)(1 - 1\!\!1_{R_n}(x))$ converges strongly to zero in $L^1(\Omega)$, because $(\nabla v_n)_n$ is equi-integrable on $\Omega$ and meas $(\Omega \setminus R_n)$ tends to zero as $n \to \infty$. Therefore the sequence $\big(\nabla v_n(x) 1\!\!1_{R_n}(x)\big)_n$ converges to the same Young measure $\nu_x^\gamma(\lambda)$ as the sequence $(\nabla v_n)_n$ (recall that $v_n = T_\gamma(u_n)$). Now since $\mathfrak{a}$ is Carathéodory and because the sequence $\big(\mathfrak{a}(x,\nabla v_n(x) 1\!\!1_{R_n}(x))\big)_n = (\bar{\chi}_n^\gamma)_n$ is already shown to be equi-integrable, we can use Theorem 2.10(i) and deduce that $\chi^\gamma(x) = \bar{\chi}^\gamma(x) = \int_{\mathbb{R}^N} \mathfrak{a}(x,\lambda)\, d\nu_x(\lambda)$ a.e. on $\Omega$.

This representation formula together with the growth assumption (4) and the result of Claim 4 imply that $\chi^\gamma \in L^{p'(\cdot)}(\Omega)$.

• Claim 8 : there exists a dense set $\mathbb{M} \subset \mathbb{R}^+$ such that the results of Claims 3-7 hold for all $\gamma \in \mathbb{M}$; moreover, for all $\gamma, \hat{\gamma} \in \mathbb{M}$ such that $\hat{\gamma} > \gamma$, $\chi^\gamma = \chi^{\hat{\gamma}} 1\!\!1_{[|u|<\gamma]}$.

With $u$ obtained in Claim 3, take an arbitrary countable set $\mathbb{M} \subset \mathbb{R}^+$ such that for all $\gamma \in \mathbb{M}$, meas $\big([|u| = \gamma]\big) = 0$. By extracting a further subsequence, we may assume that the properties of Claims 3-7 hold also with $\gamma \in \mathbb{M}$.

Now let $\gamma, \hat{\gamma} \in \mathbb{M}$ with $\hat{\gamma} > \gamma$. Let us show that $g_n := \mathfrak{a}_n(x,\nabla T_{\hat{\gamma}(u_n)}) 1\!\!1_{[|u|<\gamma]}$ converges weakly in $L^1(\Omega)$ to $\chi^\gamma$ as $n \to \infty$. Because $g_n$ also converges to $\chi^{\hat{\gamma}} 1\!\!1_{[|u|<\gamma]}$ weakly in $L^1(\Omega)$, the desired claim will follow by the uniqueness of a limit.

Since $\mathfrak{a}_n(x,0) = 0$, we have $h_n := \mathfrak{a}_n(x,\nabla T_{\hat{\gamma}}(u_n)) 1\!\!1_{[|u_n|<\gamma]} \equiv \mathfrak{a}_n(x,\nabla T_\gamma(u_n))$, so that $h_n$ converge to $\chi^\gamma$ weakly in $L^1(\Omega)$. Consider the functions

$$d_n(x) := g_n - h_n = \mathfrak{a}_n(x,\nabla T_{\hat{\gamma}}(u_n))(1\!\!1_{[|u|<\gamma]} - 1\!\!1_{[|u_n|<\gamma]}).$$

By Claim 6, the sequence $(d_n)_n$ is equi-integrable on $\Omega$. By the choice of $\mathbb{M}$, $|u| \neq \gamma$ a.e. on $\Omega$; thus we can consider that $1\!\!1_{(-\gamma,\gamma)}(\cdot)$ is continuous on the image of $\Omega$ by $u(\cdot)$. Since $u_n$ converges to $u$ a.e. on $\Omega$,

$$1\!\!1_{[|u_n|<\gamma]} = 1\!\!1_{(-\gamma,\gamma)}(u_n) \longrightarrow 1\!\!1_{(-\gamma,\gamma)}(u) = 1\!\!1_{[|u|<\gamma]} \quad \text{a.e. on } \Omega \text{ as } n \to \infty.$$

By the Vitali theorem, $d_n$ tends to zero in $L^1(\Omega)$; this $g_n = h_n + d_n$ tends to $\chi^\gamma$ in $L^1(\Omega)$ weakly. This ends the proof of Claim 8.

• Claim 9 : for all $\gamma \in \mathbb{M}$,

$$\int_\Omega \chi^\gamma \cdot \nabla T_\gamma(u) \;\geq\; \liminf_{n\to\infty} \int_\Omega \chi_n^\gamma \cdot \nabla T_\gamma(u_n). \tag{35}$$

By Proposition 3.5(i), the broad weak solution $u_n$ is also a renormalized broad solution of the same problem. Fix $e \in \mathcal{E}(\Omega)$. By the definition of $\mathcal{E}(\Omega)$, for all $n$ large enough $e$ is an admissible test function in the renormalized broad formulation (16) for $u_n$. We infer

$$\begin{aligned} \Big| \int_\Omega b(u_n)\, S'(u_n)\, e + S'(u_n)\, \chi_n^M \cdot \nabla e - f_n\, S'(u_n)\, e \Big| \\ \le \|e\|_{L^\infty} \int_\Omega |S''(u_n)|\, \chi_n^M \cdot \nabla T_M(u_n), \end{aligned} \tag{36}$$

where $\operatorname{supp} S \subset [-M, M]$, $M \in \mathbb{M}$.

Let us pass to the limit in (36). By Claim 3, $u_n$ converges to $u$ a.e. on $\Omega$. By the continuity of $b$ and $S'$, and because of the compactness of $\operatorname{supp} S'$, both terms $b(u_n)S'(u_n)$ and $S'(u_n)$ converge to $b(u)S'(u)$ and $S'(u)$, respectively, strongly in $L^1(\Omega)$. We also have

$$\int_\Omega f_n S'(u_n)\, e = \int_\Omega f_n S'(u)\, e + \int_\Omega f_n \big( S'(u_n) - S'(u) \big)\, e \longrightarrow \int_\Omega f S'(u)\, e \tag{37}$$

as $n \to \infty$, because $\displaystyle\int_\Omega f_n \big( S'(u_n) - S'(u) \big)\, e$ vanishes as $n \to \infty$. Indeed, for all $R > 0$,

$$\begin{aligned} \int_\Omega \big| f_n \big( S'(u_n) - S'(u) \big)\, e \big| \le 2\|e\|_{L^\infty} \|S'\|_{L^\infty} \int_{\big[|f_n|>R\big]} |f_n| \\ + R\, \|e\|_{L^\infty} \int_\Omega \big| S'(u_n) - S'(u) \big|. \end{aligned} \tag{38}$$

For all $R$ fixed, the second term tends to zero as $n \to \infty$. Since by the Chebyshev inequality,

$$\sup_n \operatorname{meas}\big( \big[|f_n| > R\big] \big) \;\le\; \frac{\sup_n \|f_n\|_{L^1}}{R} \;\le\; \frac{C}{R} \longrightarrow 0 \quad \text{as } R \to \infty,$$

and because a weakly convergent in $L^1(\Omega)$ sequence is equi-integrable on $\Omega$, by a choice of $R$ the first term in the right-hand side of (38) can be made as small as desired. Hence we deduce that $f_n \big( S'(u_n) - S'(u) \big)\, e$ goes to zero in $L^1(\Omega)$. Thus (37) is justified.

With a similar reasoning, we pass to the limit in the term $\displaystyle\int_\Omega S'(u_n)\, \chi_n^M \cdot \nabla e$ in (36). For $R > 0$,

$$\int_\Omega S'(u_n)\, \chi_n^M \cdot \nabla e = \int_{\big[|\nabla e|<R\big]} \chi_n^M \cdot (\nabla e\, S'(u_n)) + \int_{\big[|\nabla e|>R\big]} \chi_n^M \cdot (\nabla e\, S'(u_n)). \tag{39}$$

For all $R > 0$, by the weak $L^1(\Omega)$ convergence of $\chi_n^M$ to $\chi^M$ we get

$$\int_{\big[|\nabla e|<R\big]} \chi_n^M \cdot (\nabla e\, S'(u_n)) \to \int_{\big[|\nabla e|<R\big]} \chi^M \cdot (\nabla e\, S'(u)) \quad \text{as } n \to \infty;$$

here we have used the same argument as for (37). The second term in the right-hand side of (39) tends to zero as $R\to\infty$ uniformly in $n$. Indeed,

$$\Big|\int_{\big[|\nabla e|>R\big]}\chi_n^M\cdot(\nabla e\, S'(u_n))\Big| \le \|S'\|_{L^\infty}\,\|\chi_n^M\|_{L^{p'_n(\cdot)}}\,\|1\!\!1_{\big[|\nabla e|>R\big]}\nabla e\|_{L^{p_n(\cdot)}}. \quad (40)$$

By Claim 1 and the growth assumption (4), $\|\chi_n^M\|_{L^{p'_n(\cdot)}}\le C$. By (22) and Proposition 2.3(iii), $\sup_n\|1\!\!1_{\big[|\nabla e|>R\big]}\nabla e\|_{L^{p_n(\cdot)}}$ tends to zero as $R\to\infty$. Reproducing the decomposition (39) and the estimate (40) for the term $\int_\Omega S'(u)\,\chi^M\cdot\nabla e$, we infer

$$\begin{aligned}\Big|\int_\Omega b(u)\,S'(u)\,e+S'(u)\,\chi^M\cdot\nabla e-f\,S'(u)\,e\Big|&\le\\ \|e\|_{L^\infty}\,\sup_n\int_\Omega|S''(u_n)|\,\mathfrak{a}_n(x,\nabla T_M(u_n))\cdot\nabla T_M(u_n).&\end{aligned}\quad (41)$$

Now fix $\gamma\in\mathbb{M}$. Because $\mathcal{E}(\Omega)$ is assumed to be dense in $\dot{\mathrm{E}}^{p(\cdot)}(\Omega)$, and $T_\gamma(u)\in\dot{\mathrm{E}}^{p(\cdot)}(\Omega)\cap L^\infty(\Omega)$, we can replace $e$ by $T_\gamma(u)$ in (40).

Now we are intended to let $M\to+\infty$. One easily constructs a sequence $(S_M)_M\in\mathcal{S}$ such that

- $S'_M, S''_M$ are uniformly bounded;
- for all $M\in\mathbb{N}$, $S'_M=1$ on $[-M+1,M-1]$, $\operatorname{supp}S'\subset[-M,M]$;
- the sequence $(b(z)S'_M(z))_M$ is non-decreasing for all $z\in\mathbb{R}$.

For $M>\gamma$, thanks to Claim 8 we can replace $\chi^M\cdot\nabla T_\gamma(u)$ by $\chi^\gamma\cdot\nabla T_\gamma(u)$. Using estimates (28) and (29), and the growth assumption (4), we conclude that the right-hand side of (41) tends to zero as $M\to\infty$. Using the monotone and dominated convergence theorems in the left-hand side of (41), with $e=T_\gamma(u)$, we deduce

$$\int_\Omega\Big(\,b(u)\;T_\gamma(u)+\chi^\gamma\cdot\nabla T_\gamma(u)-f\,T_\gamma(u)\,\Big)\;=\;0. \quad (42)$$

Now, notice that $b(u)T_\gamma(u)\ge 0$; since $u_n$ converges to $u$ a.e. on $\Omega$ and by the Fatou lemma, and also because $f_n\to f$ in $L^1(\Omega)$ and $\|T_\gamma\|_{L^\infty}<+\infty$, we have

$$\int_\Omega\Big(\,b(u)\;T_\gamma(u)-f\,T_\gamma(u)\,\Big)\le\liminf_{n\to\infty}\int_\Omega\Big(\,b(u_n)\;T_\gamma(u_n)-f_n\,T_\gamma(u_n)\,\Big). \quad (43)$$

Finally, take $T_\gamma(u_n)$ as the test function in the broad formulation (13) of problem ($1_n$),(7). Comparing the so obtained equality with (42) and using (43), we infer (35).

• Claim 10 : for all $\gamma \in \mathbb{M}$, the "div-curl" inequality[6] holds:

$$\int_{\Omega\times\mathbb{R}^N} \big(\mathfrak{a}(x,\lambda) - \mathfrak{a}(x,\nabla T_\gamma(u))\big)\cdot\big(\lambda - \nabla T_\gamma(u)\big)\, d\nu_x^\gamma(\lambda)\, dx \le 0. \tag{44}$$

Starting from (35), we can deduce (44) as follows. Set $v_n := T_\gamma(u_n)$, $v := T_\gamma(u)$. By Lemma 2.1, the integral in the right-hand side of (35) is lower bounded by $\int_\Omega \mathfrak{a}_n(x, h_m(\nabla v_n))\cdot h_m(\nabla v_n)$. As in Claim 4, we use the nonlinear weak-* convergence property to get

$$\lim_{n\to\infty}\int_\Omega \mathfrak{a}_n(x,h_m(\nabla v_n))\cdot h_m(\nabla v_n) = \int_{\Omega\times\mathbb{R}^N} \mathfrak{a}(x,h_m(\lambda))\cdot h_m(\lambda)\; d\nu_x^\gamma(\lambda)\, dx.$$

As $m\to\infty$, from (35), Lemma 2.1 and the monotone convergence theorem we infer that

$$\int_\Omega \chi^\gamma\cdot\nabla T_\gamma(u) \ge \int_{\Omega\times\mathbb{R}^N} \mathfrak{a}(x,\lambda)\cdot\lambda\; d\nu_x^\gamma(\lambda)\, dx. \tag{45}$$

Now using the representation formulas (30),(33) and the fact that $\nu_x(\lambda)$ is a probability measure on $\mathbb{R}^N$ for a.e. $x\in\Omega$, we find

$$\begin{aligned}
&\int_{\Omega\times\mathbb{R}^N}\big(\mathfrak{a}(x,\lambda) - \mathfrak{a}(x,\nabla v)\big)\cdot\big(\lambda - \nabla v\big)\; d\nu_x(\lambda)\, dx\\
&\quad= \int_{\Omega\times\mathbb{R}^N} \mathfrak{a}(x,\lambda)\cdot\lambda\; d\nu_x(\lambda)\, dx - \int_\Omega\left(\int_{\mathbb{R}^N}\mathfrak{a}(x,\lambda) d\nu_x(\lambda)\right)\cdot\nabla v\; dx\\
&\qquad - \int_\Omega \mathfrak{a}(x,\nabla v)\cdot\left(\int_{\mathbb{R}^N}\lambda\; d\nu_x(\lambda)\right)\, dx + \int_\Omega\Big(\mathfrak{a}(x,\nabla v)\cdot\nabla v\Big)\left(\int_{\mathbb{R}^N} d\nu_x(\lambda)\right)\, dx\\
&\quad= \int_{\Omega\times\mathbb{R}^N} \mathfrak{a}(x,\lambda)\cdot\lambda\; d\nu_x(\lambda)\, dx - \int_\Omega\left(\int_{\mathbb{R}^N}\mathfrak{a}(x,\lambda)\, d\nu_x(\lambda)\right)\cdot\left(\int_{\mathbb{R}^N}\lambda\, d\nu_x(\lambda)\right) dx.
\end{aligned}$$

By (45), using (30),(33) again, we infer (44).

• Claim 11 : for all $\gamma\in\mathbb{M}$,

$$\chi^\gamma(x) = \mathfrak{a}(x,\nabla T_\gamma(u(x))) \quad \text{for a.e. } x\in\Omega, \tag{46}$$

and $\nabla T_\gamma(u_n)$ converges to $\nabla T_\gamma(u)$ in measure on $\Omega$, as $n\to\infty$.

Indeed, by (44) and the strict monotonicity assumption (3) on $\mathfrak{a}(x,\cdot)$, for a.e. $x\in\Omega$ we have $\lambda = \nabla T_\gamma(u(x))$ a.e. wrt the measure $\nu_x^\gamma$ on $\mathbb{R}^N$. Therefore, the measure $\nu_x^\gamma$ reduces to the Dirac measure $\delta_{\nabla T_\gamma(u(x))}$. Now (46) follows from (33). Moreover, by Theorem 2.10(ii), we deduce that $\nabla T_\gamma(u_n) \Rightarrow \nabla T_\gamma(u)$.

• Claim 12 : for all $\gamma\in\mathbb{M}$, $\chi_n^\gamma\cdot\nabla T_\gamma(u_n) \longrightarrow \chi^\gamma\cdot\nabla T_\gamma(u)$ strongly in $L^1(\Omega)$.

[6]this terminology was proposed by Dolzmann, Hungerbühler and Müller (see [27] and [41]); it underlines the "compensated compactness" nature of the monotonicity argument used in this Claim

Indeed, by the previous claim and because of assumption (19), extracting a further subsequence we can assume that the sequence $(G_n)_n$, $G_n := \chi_n^\gamma \cdot \nabla T_\gamma(u_n) \equiv \mathfrak{a}_n(x, \nabla T_\gamma(u_n)) \cdot \nabla T_\gamma(u_n)$, converges a.e. on $\Omega$ to the function $G$ defined as $G := \chi^\gamma \cdot \nabla T_\gamma(u) \equiv \mathfrak{a}(x, \nabla T_\gamma(u)) \cdot \nabla T_\gamma(u)$. Because $G_n \geq 0$, by the Fatou lemma we infer $\int_\Omega G \leq \liminf_{n\to\infty} \int_\Omega G_n$. Because (35) asserts that the inequality "$\geq$" is true, we conclude that for the sequence $(G_n)_n$, the Fatou lemma holds with the equality sign. Hence the $L^1$ convergence of (a subsequence of) $(G_n)_n$ to $G$ follows.[7]

• Claim 13 : $u$ is a renormalized broad solution of (1),(7).

First, let us deduce the constraint (14). By the growth assumption (4) and thanks to the estimate (29), the constraint (14) follows from property (32) shown in Claim 5.

The other requirements in Definition 3.3 being trivially satisfied, it remains to show (16); because $\mathcal{E}(\Omega)$ is dense in $\dot{\mathrm{E}}^{p(\cdot)}(\Omega)$ by assumption (22), it suffices to show (16) with a test function in $\mathcal{E}(\Omega)$. We repeat the reasoning that led to (41); but now the term $\int_\Omega S''(u_n)\, \chi_n^M \cdot \nabla T_M(u_n)\, e$ should be examined. Thanks to the previous claim, and because $S''(u_n) \to S(u)$ a.e. on $\Omega$ and remains bounded, we deduce the renormalized formulation (16) for all test function in $\mathcal{E}(\Omega)$. This ends the proof of our claim.

• Claim 14 : The whole sequence $(u_n)_n$ converges to $u$ a.e on $\Omega$ as $n \to \infty$. Moreover, the whole sequence $(\nabla u_n)_n$ converges to $v$ a.e. on $\Omega$ , where $v$ is defined by formula (18).

Indeed, recall the result of Claim 1. First note that $v$ is well defined (see the proof of Proposition 3.5(ii)). By Claim 11 and because $u$ is finite a.e. on $\Omega$ (see Claim 3), we deduce that $\nabla u_n$ converges to $v$ a.e. on $\Omega$, up to extraction of a subsequence. Now, by Claim 13 and the uniqueness of a renormalized solution to (1),(7) asserted in Theorem 3.12, we conclude that all convergent subsequences of $(u_n)_n$, $(\nabla u_n)_n$ converge to the same limits $u$, $v$, respectively.

• Claim 15 : because $f \in \left(\dot{\mathrm{E}}^{p(\cdot)}(\Omega)\right)^*$, $v = \nabla u$ in the sense of distributions, and $u$ is in fact a weak solution of (1),(7).

This follows from Proposition 3.5(ii); we only have to prove that $v \in L^{p(\cdot)}(\Omega)$. By the Fatou lemma and the definition (18) of $v$, it is sufficient to show that $\rho_{p(\cdot)}(\nabla T_\gamma(u)) \equiv \int_\Omega |\nabla T_\gamma(u)|^{p(x)} \leq C$, where $C$ is independent of $\gamma$. To this end, take $T_\gamma(u)$ as the test function in the renormalized formulation (16). Choose $S \in \mathcal{S}_0$ satisfying $S'' \equiv 0$ on $[-\gamma, \gamma]$, $S' \equiv 1$ on $[-\gamma, \gamma]$, and $\|S'\|_{L^\infty} \leq 1$.

[7] This complement of the Fatou lemma is well known in the probability theory, where it is called the Scheffé's theorem (see [64]). It is also an easy case of the Brézis-Lieb lemma ([19]).

Dropping the nonnegative term $S'(u)b(u)T_\gamma(u)$, we infer

$$\int_\Omega \mathfrak{a}(x,\nabla T_\gamma(u))\cdot\nabla T_\gamma(u) \le \int_\Omega S'(u)\, f\, T_\gamma(u) \le \|f\|_{\left(\dot{\mathrm{E}}^{p(\cdot)}\right)^*}\, \|T_\gamma(u)\|_{\dot{\mathrm{E}}^{p(\cdot)}}.$$

By the coercivity assumption (5) and Proposition 2.3(iii) we deduce a bound of the modular $\rho_{p(\cdot)}(\nabla T_\gamma(u))$ and on $\|T_\gamma(u)\|_{\dot{\mathrm{E}}^{p(\cdot)}}$ that does not depend on $\gamma$.

This ends the proof of Theorem 3.8. $\diamond$

Proof of Theorem 3.7: We only indicate the changes with respect to the proof of Theorem 3.8.

(i) This is straightforward. According to Remark 3.9, we pick $\mathcal{E}(\Omega)=\dot{\mathrm{E}}^{p(\cdot)}(\Omega)$.

(ii) The difference with the above proof appears in Claim 9. Here we use $\mathcal{D}(\Omega)$ in the place of the set $\mathcal{E}(\Omega)$. Therefore, in order to replace the test function $e$ in (41) with the function $T_\gamma(u)$, we need that $T_\gamma(u)$ belong to the closure of $\mathcal{D}(\Omega)$ in the norm of $\dot{\mathrm{E}}^{p(\cdot)}(\Omega)$. That is, we need $T_\gamma(u)\in W_0^{1,p(\cdot)}(\Omega)$. This property is enforced by the assumption that $p_n\ge p$ a.e., and the fact that $u_n$ are themselves narrow weak solutions to problems ($1_n$),(7).

Indeed, by Lemma 2.9, since $u_n\in W_0^{1,p_n(\cdot)}(\Omega)$ we also have $T_\gamma(u_n)\in W_0^{1,p_n(\cdot)}(\Omega)\subset W_0^{1,p(\cdot)}(\Omega)$; moreover, $\big(T_\gamma(u_n)\big)_n$ is bounded in $W_0^{1,p(\cdot)}(\Omega)$ by (26) and because $p_n\ge p$. Thus, we can add to Claim 3 the fact that $T_\gamma(u_n)$ converges to $T_\gamma(u)$ also in $W_0^{1,p(\cdot)}(\Omega)$ weakly.

Further, in Claim 13, we can assert that $u$ is a renormalized narrow solution of (1),(7). Indeed, we are only allowed to take $e\in\mathcal{D}(\Omega)$, but this is enough to deduce (15).

Finally, the a.e. convergence of $\nabla u_n$ to $\nabla u$ in Claim 14 can be upgraded to the strong convergence of $T_\gamma(u_n)$ to $T_\gamma(u)$ in $W_0^{1,p(\cdot)}(\Omega)$. Indeed, by Claim 12, for a dense set $\mathbb{M}$ of values of $\gamma$, the sequence $\big(\mathfrak{a}_n(x,\nabla T_\gamma(u_n))\cdot\nabla T_\gamma(u_n)\big)_n$ is equi-integrable on $\Omega$. By the coercivity assumption (5) and because $p_n\ge p$, the sequence $\big(|\nabla T_\gamma(u_n)|^{p(x)}\big)_n$ is equi-integrable. By the Vitali theorem, $\nabla T_\gamma(u_n)$ tends to $\nabla T_\gamma(u)$ in $L^{p(\cdot)}(\Omega)$. One easily extends this result to all values of $\gamma$ with the help of the technique used at the end of the proof of Lemma 2.9. $\diamond$

**Remark 4.1.** In Theorem 3.7(ii), under stronger hypotheses on $(f_n)_n$, the $W_0^{1,p(\cdot)}$ convergence of $u_n$ can be asserted. In particular, if assumption (21) is replaced by the assumption

$$(f_n)_n\subset L^{(p^*(\cdot))'}(\Omega),\ f_n \text{ converges to } f\in L^{(p^*(\cdot))'}(\Omega) \text{ weakly in } L^{(p^*(\cdot))'}(\Omega), \tag{47}$$

and if $p(\cdot)$ satisfies (11), then $u_n$ converges to $u$ in $W_0^{1,p(\cdot)}(\Omega)$ strongly.

Proof : Notice that Proposition 2.3(iv) allows to use the Poincaré inequality in $W_0^{1,p(\cdot)}(\Omega)$. Because $p_n\ge p$, under assumption (47) we have a uniform estimate of $\|f_n\|_{L^{p_n(\cdot)}}$; thus the estimates on $T_\gamma(u_n)$ are uniform in $\gamma$, and we can avoid the use of truncations in Claim 1 of the proof of Theorem 3.8. The sequence $(u_n)_n$ is then bounded in $W_0^{1,p(\cdot)}(\Omega)$. In addition, thanks to the optimal injection result of Proposition 2.4(ii), we get the equality $\lim\limits_{n\to\infty}\int_\Omega f_n u_n = \int_\Omega f u$. These two facts allow us to deduce Claims 3,4,6,7, and 9—12 in the proof of Theorem 3.8 with $\gamma=+\infty$. In particular, using Claim 12 in the same way as at the end of the proof of Theorem 3.7(ii), we get the strong convergence of $\nabla u_n$ to $\nabla u$ in $L^{p(\cdot)}(\Omega)$. $\diamond$

Proof of Theorem 3.10: The proof is essentially contained in the proofs of Theorems 3.8,3.7.

(i) We only have to use the renormalized broad formulation of ($1_n$),(7) in order to obtain the properties (26),(28),(29) in Claims 1,2; these estimates are standard in the context of renormalized solutions. The rest of the proof of Theorem 3.8 applies without changes, except for Claim 15.

(ii) Instead of Theorem 3.8, we refer to Theorem 3.7. $\diamond$

## 5. Existence and uniqueness of solutions to (1),(7): the $p(x)$-case

Proof of Theorem 3.11: Let us focus on the case of narrow solutions.

• Step 1. We show existence of narrow weak solutions for an $L^\infty$ source term $f$, by constructing a sequence of Galerkin approximations. Pick a countable set $(w_i)_i$ which spans the Banach space $W_0^{1,p(\cdot)}(\Omega)$. For $n\in\mathbb{N}$ and $(c_i^n)_{i=1}^n\subset\mathbb{R}$, define $u_n(x):=\sum_{i=1}^n c_i^n\, w_i(x)$. Notice that the sub-linearity of $\mathcal{L}$ and the optimal injection result of Proposition 2.4(ii) for $W_0^{1,r(\cdot)}(\Omega)$, where $r\in\mathcal{R}(\Omega)$, $r\le p$ a.e. on $\Omega$, imply that

$$\int_\Omega \mathcal{L}\big(\, u_n\, b(u_n)+|u_n|^{r^*(x)}\,\big)\le C(\varepsilon)+\varepsilon\int_\Omega\big(\, u_n\, b(u_n)+|\,\nabla u_n|^{p(x)}\,\big) \tag{48}$$

for all $\varepsilon>0$. Combining (48) with the coercivity assumption (24), we see that

$$\begin{aligned}\int_\Omega\big(\, b(u_n)\, u_n+\mathfrak{a}(x,u_n,\nabla u_n)\cdot\,\nabla u_n - f\, u_n\,\big)&\\ \ge\ \frac{1}{2C}\int_\Omega\big(\, u_n\, b(u_n)+|\nabla u_n|^{p(x)}\big)-C(1+\|f\|_{L^\infty}).&\end{aligned} \tag{49}$$

By a standard application of the Brouwer fixed-point theorem (see [45, Ch.I,Lemma 4.3]) we deduce that there exists a solution to the nonlinear system on $(c_i^n)_{i=1}^n$:

$$u_n(x):=\sum_{i=1}^n c_i^n\, w_i(x),\quad \int_\Omega\big(\, b(u_n)\, w_i+\mathfrak{a}(x,u_n,\nabla u_n)\cdot\nabla w_i\,\big)=\int_\Omega f\, w_i,\ \ i=1,\dots,n.$$

Moreover, thanks to (49), the sequence $(u_n)_n$ is bounded in $W_0^{1,p(\cdot)}(\Omega)$, and the sequence $\big(\, u_n\, b(u_n)\,\big)_n$ is bounded in $L^1(\Omega)$. We denote by $u$ its weak accumulation point in $W_0^{1,p(\cdot)}(\Omega)$; we also have (for a subsequence) $u_n\to u$ a.e. on $\Omega$. Now Claims 3,4 of the proof of Theorem 3.8 apply.

Further, using again the sub-linearity of $\mathcal{L}$, from the above bounds and the growth assumption (23) we deduce that $\tilde{\mathfrak{a}}_n(x,\xi) := \mathfrak{a}(x, u_n(x), \xi)$ verifies

$$|\tilde{\mathfrak{a}}_n(x,\xi)|^{p'(x)} \le C\left(|\xi|^{p(x)} + \mathcal{M}_n(x)\right), \tag{50}$$

with an equi-integrable sequence of functions $\left(\mathcal{M}_n\right)_n$ in $L^1(\Omega)$. Let us show that (up to extraction of a subsequence) $\tilde{\mathfrak{a}}_n$ converge to $\tilde{\mathfrak{a}}$, $\tilde{\mathfrak{a}}(x,\xi) := \mathfrak{a}(x,u(x),\xi)$ in the sense (19), i.e.,

$$\left|\begin{array}{l} \text{for all bounded subset } K \text{ of } \mathbb{R}^N, \\ \sup_{\xi\in K} |\tilde{\mathfrak{a}}_n(\cdot,\xi) - \tilde{\mathfrak{a}}(\cdot,\xi)| \text{ converges to zero in measure on } \Omega. \end{array}\right. \tag{51}$$

For the proof, consider the function $(x,\xi) \mapsto \mathfrak{a}(x,u(x),\xi)$ as $x \mapsto \mathfrak{a}(x,u(x),\cdot)$, a mapping from $\Omega$ to $C(\mathbb{R}^N)$ (supplied with the topology of locally uniform convergence). Define the maps $x \mapsto \mathfrak{a}_n(x,\cdot) =: \mathfrak{a}(x,u_n(x),\cdot)$ analogously. We will apply the Egorov theorem; for the sake of completeness, let us justify the fact that the so defined maps are measurable.

For a measure $\mu \in (C(\mathbb{R}\times\mathbb{R}^N))^*$, consider the function

$$g_\mu : x \in \Omega \ \mapsto < \mu\,,\, \mathfrak{a}(x,u(x),\cdot)\ >_{\left(C(\mathbb{R}\times\mathbb{R}^N)\right)^*, C(\mathbb{R}\times\mathbb{R}^N)}.$$

For all fixed $\xi_0 \in \mathbb{R}^N$, consider the Dirac measure $\delta_{\xi_0} \in \left(C(\mathbb{R}^N)\right)^*$; then $g_{\delta_{\xi_0}}(\cdot) = \mathfrak{a}(\cdot,u(\cdot),\xi_0)$ is measurable, because $u$ is measurable and $\mathfrak{a}$ is Carathéodory. Because all measure $\mu$ can be approximated by a weakly convergent sequence $(\mu_k)_k$ of finite sums of Dirac measures, $g_\mu$ is the pointwise limit of measurable functions $g_{\mu_k}$. We conclude that the map $x \mapsto \mathfrak{a}(x,u(x),\cdot)$ is weakly measurable. Hence it is strongly measurable (e.g., cf. [17, Chap.IV,§5,Prop.10]).

Since $u_n$ converges to $u$ a.e. on $\Omega$ and because $\mathfrak{a}(x,z,\xi)$ is continuous in $(z,\xi)$, we deduce that $\tilde{\mathfrak{a}}_n(x,\cdot)$ converges to $\tilde{\mathfrak{a}}(x,\cdot)$ in $C(\mathbb{R}^N)$, for a.e. $x \in \Omega$. Applying the Egorov theorem, we conclude that the convergence is uniform on the complementary of an open set $E_\alpha \subset \Omega$, $\text{meas}\,(\Omega \setminus E_\alpha) < \alpha$. Thus for all compact subset $K$ of $\mathbb{R}^N$, $\tilde{\mathfrak{a}}_n(x,\xi)$ converges to $\tilde{\mathfrak{a}}_n(x,\xi)$ uniformly in $(x,\xi) \in \left(\Omega \setminus E_\alpha\right) \times K$. This implies (51).

With (50), (51) in hand, we can apply Claims 6,7 of the proof of Theorem 3.8, where we can formally put $\gamma = \infty$. Then we reason as for the proof of Theorem 3.7(ii)[8], thanks to the fact that $p(x)$ does not change with $n$. We conclude that $u$ is a narrow weak solution of the Dirichlet problem of the form $b(u) - \text{div}\,\tilde{\mathfrak{a}}(x,\nabla u) = f$; recalling that $\tilde{\mathfrak{a}}(x,\nabla u) = \mathfrak{a}(x,u(x),\nabla u)$, we conclude the existence proof for an $L^\infty$ source term $f$.

• Step 2. Now we can deduce the claims of Theorem 3.11 by applying the stability results of Theorems 3.7,3.10 (we only have to modify the part of

[8]in this case, the proof of the key inequality (35) goes with many simplifications, due to the fact that we can put $\gamma = \infty$ and because the convergence of $\displaystyle\int_\Omega f\,u_n$ to $\displaystyle\int_\Omega f\,u$ is trivial.

the proof devoted to the *a priori* estimates, in order to take into account assumptions (23),(24) for $\mathfrak{a}$, more general than those allowed in the statements of Theorems 3.7,3.10).

For $f \in L^1(\Omega)$, consider the sequence of truncations $(T_n(f))_n \subset L^\infty(\Omega)$. By the result of Step 1, we can construct a sequence of the associated narrow weak solutions $(u_n)_n$. At this point, we need to take into account that, thanks to (2), $\mathfrak{a}(x, u, \nabla T_\gamma(u)) \equiv \mathfrak{a}(x, T_\gamma(u), \nabla T_\gamma(u))$; also $T_\gamma(z) b(T_\gamma(z)) \le T_\gamma(z) b(z)$ for all $z \in \mathbb{R}$. Therefore assumption (23) yields a uniform in $n$ estimate of the form

$$\int_\Omega \big( T_\gamma(u_n)\, b(T_\gamma(u_n)) + |\nabla T_\gamma(u_n)|^{p(x)} \big) \;\le\; C(\gamma).$$

Then, because $p(x)$ is independent of $n$, we can apply the convergence argument of Theorem 3.10(ii); to this end, we consider $\mathfrak{a}(x, u_n(x), \xi)$ as $\tilde{\mathfrak{a}}_n(x, \xi)$ and deduce (50),(51) in the way it is done in Step 1. This justifies the existence of a renormalized narrow solution.

If, in addition, $f \in W_0^{-1,p'(\cdot)}(\Omega)$, then instead of Theorem 3.10(ii) we refer to Theorem 3.7(ii). This justifies the existence of a narrow weak solution.

• Finally, the proofs of existence for broad solutions are entirely similar. We only have to pick the Galerkin basis $(w_i)_i$ accordingly to the larger space $\dot{\mathrm{E}}^{p(\cdot)}(\Omega)$ in the first step of the proof. $\diamond$

Proof of Theorem 3.12 (sketched): For the proof, the $L^1$ techniques are used. The argument is well-known in the context of problems of the kind (1),(7) with a constant exponent $p$, and it runs without changes when the exponent is variable. We give it for the sake of completeness.

Write $u, \hat{u}$ for $u_f, u_{\hat{f}}$, respectively. When both $u, \hat{u}$ are narrow weak solutions, the test function $\phi := \frac{1}{\gamma} T_\gamma(u - \hat{u})^+$ is admissible in the $\mathcal{D}'(\Omega)$ formulation of (1),(7), thanks to Lemma 2.9 and the standard density argument. When both $u, \hat{u}$ are broad weak solutions, $\phi$ is an admissible test function in (13). By the monotonicity hypothesis (3), we infer

$$\int_\Omega (b(u) - b(\hat{u}))\, \frac{1}{\gamma} T_\gamma(u - \hat{u})^+ \le \int_\Omega (f - \hat{f})\, \frac{1}{\gamma} T_\gamma(u - \hat{u})^+ \tag{52}$$

As $\gamma \to 0$, inequality (25) follows. If $b$ is strictly increasing, then uniqueness is immediate.

If $b$ is not strictly increasing and $f = \hat{f}$, then in the above inequality (52) we replace $T_\gamma(u - \hat{u})^+$ by $T_\gamma(u - \hat{u})$; moreover, we keep the term

$$\int_{\left[0 < u - \hat{u} < \gamma\right]} \big(\mathfrak{a}(x, \nabla u) - \mathfrak{a}(x, \nabla \hat{u})\big) \cdot \big(\nabla u - \nabla \hat{u}\big)$$

in the left-hand side. From the strict monotonicity assumption (3) on $\mathfrak{a}$ we deduce that $\nabla u = \nabla \hat{u}$ a.e. on $\left[0 < u - \hat{u} < \gamma\right]$. Because $\gamma$ is arbitrary, as $\gamma \to \infty$ we conclude that $\nabla u = \nabla \hat{u}$ a.e. on $\Omega$. By the Poincaré inequality in $W^{1,p_-}(\Omega)$, we infer that $u = \hat{u}$.

For the case where $u, \hat{u}$ are renormalized solutions, (52) is not straightforward. Using the renormalized formulations (15) or (16), we first get

$$\begin{aligned}
\int_\Omega \big( S'_M(u)b(u) - S'_M(\hat{u}))b(\hat{u}) \big)\, \phi \\
+ \int_\Omega (S'_M(u)\mathfrak{a}(x,\nabla u) - S'_M(\hat{u})\mathfrak{a}(x,\nabla \hat{u})) \cdot \nabla \phi \\
+ \int_\Omega \big( S''_M(u)\mathfrak{a}(x,\nabla u) \cdot \nabla u - S''_M(\hat{u})\mathfrak{a}(x,\nabla \hat{u}\,\big) \cdot \nabla \hat{u})\, \phi \\
\leq \int_\Omega (S'_M(u) f - S'_M(\hat{u})\hat{f})\, \phi
\end{aligned} \tag{53}$$

with $\phi = \frac{1}{\gamma} T_\gamma(T_k(u) - \hat{u})^+$. Here $(S_M)_M$, $M \to \infty$, is a sequence of functions in $\mathcal{S}$ such that $S'_M, S''_M$ are uniformly bounded, and, in addition, $S'_M = 1$ on $[-M+1, M-1]$, $\operatorname{supp} S' \subset [-M, M]$. Notice that for $k < +\infty$, $\frac{1}{\gamma} T_\gamma(T_k(u) - \hat{u})^+$ is indeed an admissible test function in (15) or (16). While $M$ is fixed, $k$ can be sent to infinity. Now with $M \to +\infty$, using the constraint (14), we see that the third term in (53) converges to zero. The first term in (53) and the last one converge, respectively, to $\displaystyle\int_\Omega (b(u) - b(\hat{u}))\, \phi$ and $\displaystyle\int_\Omega (f - \hat{f})\, \phi$, by the dominated convergence theorem. Finally, by (3), the second term in (53) is lower bounded by

$$-\frac{1}{\gamma} \int_{\left[0<u-\hat{u}<\gamma\right]} \big|\, S'_M(u) - S'_M(\hat{u}) \,\big|\, |\mathfrak{a}(x,\nabla u)| \, \big( |\nabla u| + |\nabla \hat{u}| \,\big). \tag{54}$$

Because the factor $\big|\, S'_M(u) - S'_M(\hat{u}) \,\big|$ is supported on $\big[M < |u| < M+1\big] \cup \big[M < |\hat{u}| < M+1\big]$, the whole term is supported on $\big[M-\gamma < |u| < M+\gamma+1\big] \cap \big[M-\gamma < |\hat{u}| < M+\gamma+1\big]$. Using the constraint (14), the growth assumption (4), and the properties (ii),(iii) of Proposition 2.3, we deduce that the lower bound (54) converges to zero as $M \to \infty$. Therefore (53) yields (52), at the limit $M \to \infty$. This proves (25) also in the case where $u, \hat{u}$ are renormalized solutions of the same kind, broad or narrow.

The proof of the uniqueness of renormalized solutions in the case $b$ is not strictly increasing is a combination of the above arguments (see e.g. [51]). Notice that because $u, \hat{u}$ are finite a.e. on $\Omega$, equality $\nabla u = \nabla \hat{u}$ a.e. on $\Omega$ still yields $u = \hat{u}$ a.e. on $\Omega$. $\diamond$

## Appendix: Are broad and narrow weak solutions indeed different?

In the paper [63], the relevancy of the notions of solution of type I and II (which correspond to the notions of narrow and broad solutions of the present paper), in the variational setting, was illustrated in terms of the dual minimization problem.

In order to further stress the fact that both the narrow and broad solutions should be considered simultaneously, let us indicate the following result. Consider the case $\mathfrak{a}(x,\xi) = p(x)(1 + |\xi|^2)^{\frac{p(x)-2}{2}} \xi$ with $2 \leq p(\cdot)$; take $b = \mathrm{Id}$ and fix the source term in a suitable way. Assume $p : \Omega \longrightarrow [p_-, p_+]$ is measurable; when $p_- \geq 2$, $\mathfrak{a}$ satisfies assumptions (3)-(5). Take $f \in L^{((p_-)^*)'}(\Omega)$. Consider a flow of exponents $\theta \in \mathbb{R} \mapsto p_\theta(\cdot)$ such that $p_\theta : \Omega \longrightarrow [p_-, p_+]$ is measurable. By the standard variational technique, one easily shows that for all $\theta \in \mathbb{R}$, there exists a unique minimizer $u_\theta^{narr}$ to the functional

$$J_\theta : v \mapsto \int_\Omega \Big( \frac{1}{2} v^2 + (1 + |\nabla v|^2)^{\frac{p_\theta(x)}{2}} - f\, v \Big) \tag{55}$$

on $W_0^{1,p(\cdot)}(\Omega)$, and $u_\theta^{narr}$ is the unique narrow weak solution of the problem

$$u - \operatorname{div} \Big( p_\theta(x)(1 + |\nabla u|^2)^{\frac{p_\theta(x)-2}{2}} \nabla u \Big) = f(x), \quad u|_{\partial\Omega} = 0. \tag{56}$$

Similarly, for all $\theta \in \mathbb{R}$, there exists a unique minimizer $u_\theta^{br}$ to the functional $J_\theta$ on $\dot{\mathrm{E}}^{p(\cdot)}(\Omega)$, which is the unique broad weak solution of problem (56). We have the following observation[9].

**Proposition 5.1.** *In the above setting, assume in addition that*
*- for all $\theta \neq 0$, $p_\theta : \Omega \longrightarrow [p_-, p_+]$ satisfies (11);*
*- for a.e. $x \in \Omega$, the map $\theta \mapsto p_\theta(x)$ is non-decreasing;*
*- $p_0$ coincides with $p$, and the flow $\theta \mapsto p_\theta(\cdot)$ is continuous in measure on $\Omega$.*

*Consider the maps $\theta \in \mathbb{R} \mapsto J_\theta(u_\theta^{narr})$ and $\theta \in \mathbb{R} \mapsto J_\theta(u_\theta^{br})$. The two maps coincide with a non-decreasing continuous function $j$ on $\mathbb{R} \setminus \{0\}$; moreover, $J_0(u_0^{narr}) = j(0^+)$, and $J_0(u_0^{br}) = j(0^-)$, where $j(0^\pm)$ denote the one-sided limits of $j$ at the point zero. In addition, the broad weak solution $u_0^{br}$ coincides with the narrow weak solution $u_0^{narr}$ of problem (56) if and only if the function $j$ turns out to be continuous at $\theta = 0$.*

Proof : The monotonicity in $\theta$ of the energy critical level functions $J_\theta(u_\theta^{narr})$ and $J_\theta(u_\theta^{br})$ is straightforward from the fact that the functional $J_\theta(v)$ is monotone non-decreasing in $\theta$, for all $v$. Because for $\theta \neq 0$, broad and weak solutions coincide by Remark 3.2, we only have to justify that functions

$$\theta \in \mathbb{R} \mapsto J_\theta(u_\theta^{narr}) \quad \text{and} \quad \theta \in \mathbb{R} \mapsto J_\theta(u_\theta^{br})$$

are, respectively, right-continuous and left-continuous. Let us show that $J_{\theta_n}(u_{\theta_n}^{br})$ converges to $J_\theta(u_\theta^{br})$ as $\theta_n \uparrow \theta$; and $J_{\theta_n}(u_{\theta_n}^{narr})$ converges to $J_\theta(u_\theta^{narr})$ as $\theta_n \downarrow \theta$. We refer to the proof of Theorem 3.7; $u_n$ will stand for $u_{\theta_n}^{br}$ or for $u_{\theta_n}^{narr}$, according to the monotonicity of $(\theta_n)_n$.

Recall the fact that $\mathfrak{a}_{\theta_n}(x,\nabla u_{\theta_n}) \cdot \nabla u_{\theta_n}$ converges to $\mathfrak{a}(x,\nabla u) \cdot \nabla u$ strongly in $L^1(\Omega)$, with an obvious meaning of notation (this is Claim 12 of the proof of

[9] similar properties hold for the standard $p_\theta(x)$-laplacian operators. In this case, the energy $j(\theta)$ of the minimizer of $J_\theta$ in Proposition 5.1 is not necessarily a monotone function of $\theta$; but it is continuous at $\theta \neq 0$, and the possible jump at $\theta = 0$ corresponds to the difference of the levels of energy $J_0$ of the narrow and the broad solution.

Theorem 3.8; under the assumptions we take for $f$, we can put $\gamma = +\infty$). Therefore this sequence is equi-integrable; hence the sequence $\left( (1+|\nabla u_{\theta_n}|^2)^{\frac{p_{\theta_n}(x)}{2}} \right)_n$ is equi-integrable. Because $\nabla u_{\theta_n}$ converges to $\nabla u_\theta$ a.e. on $\Omega$, we can apply the Vitali theorem. Thus the claim follows. $\diamond$

**Remark 5.2.** Notice that a map $\theta \mapsto p_\theta(\cdot)$ verifying the properties of Proposition 5.1 does not necessarily exist. For instance, take $\Omega = (0,1)$ and consider a Cantor set $A \subset (0,1)$ of positive measure. Set $p = 3 - 1\!\!1_A$. Clearly, for all continuous function $\pi$ on $(0,1)$ such that $\pi \geq p$ a.e., we have $\pi \geq 3$. Therefore $p$ cannot be approximated in measure by a decreasing sequence of continuous functions.

On the other hand, for $L > 0$ (the adaptation for $L < 0$ is straightforward) we can define

$$p_L := \inf\left\{ \, \pi \in C(\overline{\Omega}) \; \middle| \; \pi(x) \geq p(x) \text{ for a.e. } x \in \Omega, \text{ and } \pi \text{ satisfies (11)} \, \right\}. \quad (57)$$

By the Arzela-Ascoli theorem, $p_L$ verifies (11). By construction, $p_L$ does not increase with $L$. Moreover, definition (57) implies that for all $L, M > 0$, $\frac{1}{2}(p_L + p_M) \geq p_{\frac{L+M}{2}}$. Hence the map $L \in (0,+\infty) \mapsto p_L \in C(\overline{\Omega})$ is convex and non-increasing. Thus the map $\theta \in (0,+\infty) \mapsto p_{\frac{1}{\theta}}$ is continuous in measure on $\Omega$ and non-decreasing. Assuming some mild regularity of $p(\cdot)$, we can also assert that $p_\theta$ converges to $p$ in measure on $\Omega$, as $\theta \to 0$; this is true e.g. for the discontinuous, piecewise constant exponent $p$ featuring in the Zhikov's example of non-density of $\mathcal{C}^\infty(\overline{\Omega})$ in $W^{1,p(\cdot)}(\Omega)$ (see [63, 64, 67]).

**Remark 5.3.** We guess that the "gap" between $W_0^{1,p(\cdot)}(\Omega)$ and $\dot{\mathrm{E}}^{p(\cdot)}(\Omega)$ can possibly result in a gap between $j(0^+)$ and $j(0^-)$ and thus, in the non-coincidence between the narrow and the broad solutions of problems of the $p(x)$-laplacian kind. This gap could even produce a variety of "intermediate" weak solutions of the $p(x)$-laplacian which are the minimizers of $J_0$ on subspaces $E$ of $\dot{\mathrm{E}}^{p(\cdot)}(\Omega)$, $E \supset W_0^{1,p(\cdot)}(\Omega)$. One example is $E = \dot{W}^{1,p(x)}(\Omega)$. Such intermediate solutions would correspond to values of $J_0$ intermediate between $J_0(u_0^{br}) = j(0^-)$ and $J_0(u_0^{narr}) = j(0^+)$.

Let us point out that we do not know whether the narrow and broad solutions of (1),(7) can be indeed different for $L^1$ source terms $f$[10]. But starting from any $u \in \dot{\mathrm{E}}^{p(\cdot)}(\Omega) \setminus W_0^{1,p(\cdot)}(\Omega)$, it is easy to construct $f_u \in \left(\dot{\mathrm{E}}^{p(\cdot)}(\Omega)\right)^* \subset W^{-1,p(\cdot)'}(\Omega)$ such that the minimizers of, e.g., the functional

$$J : v \mapsto \int_\Omega |\nabla v|^{p(x)} - \; < f_u \, , \, v >$$

in $\dot{\mathrm{E}}^{p(\cdot)}(\Omega)$ and in $W_0^{1,p(\cdot)}(\Omega)$ are different. Indeed, it suffices to take $f_u$ in the

[10] a partial negative answer to this question is given in (58) below

subdifferential (evaluated at the point $u$) of the functional $v \mapsto \int_\Omega |\nabla v|^{p(x)}$; then $u$ is the minimizer in $\dot{\mathrm{E}}^{p(\cdot)}(\Omega)$, but $u \notin W_0^{1p(\cdot)}(\Omega)$.

Now, we also claim that for merely continuous variable exponents $p$ on $\Omega$, if a distinction between narrow and broad solutions of problem (56) actually occurs, it remains an exceptional event. To this end, consider a flow $\theta \mapsto p_\theta(\cdot)$ which is increasing and continuous in measure on $\Omega$. For instance, we can take $p_\theta(x) = p(x) + \theta$ in a neighbourhood of $\theta = 0$, with some fixed continuous function $p : \overline{\Omega} \mapsto [p_-, p_+]$. Fix $f$ and consider the energy (55). Denote by $j^{br,narr}$ the functions $\theta \mapsto J_\theta(u_{f,\theta}^{br,narr})$, respectively, where $u_{f,\theta}^{br}$,$u_{f,\theta}^{narr}$ are the minimizers of $J_\theta$ over $\dot{\mathrm{E}}^{p_\theta(\cdot)}(\Omega)$ and over $W_0^{1,p_\theta(\cdot)}(\Omega)$, respectively. As in Proposition 5.1, we deduce that

(a) both $j^{narr}$ and $j^{br}$ are non-decreasing;
(b) $j^{narr}$ is right-continuous, and $j^{br}$ is left-continuous;
(c) $j^{narr}(\theta) \geq j^{br}(\theta)$ for all $\theta$.

Let us justify one more property:

(d) if $\theta < \hat{\theta}$, then $j^{narr}(\theta) \leq j^{br}(\hat{\theta})$.

Property (d) holds if, for all $\varepsilon > 0$, $\dot{\mathrm{E}}^{p(\cdot)+\varepsilon}(\Omega) \subset W_0^{1,p(\cdot)}(\Omega)$. In our case, there exists $r \in C^\infty(\overline{\Omega})$ satisfying $p \leq r \leq p + \varepsilon$. Such regular exponent $r$ can be constructed by the usual mollifier techniques starting from $p + \varepsilon/2$, with a mollification parameter controlled by the modulus of continuity of $p$. With this construction, we do have $\dot{\mathrm{E}}^{p(\cdot)+\varepsilon}(\Omega) \subset \dot{\mathrm{E}}^{r(\cdot)}(\Omega) \equiv W_0^{1,r(\cdot)}(\Omega) \subset W_0^{1,p(\cdot)}(\Omega)$ due to Corollary 2.6.

From the above properties (a)-(d), it follows that $j^{narr}$ and $j^{br}$ are the right-continuous and the left-continuous representative, respectively, of some non-decreasing function $j$. In particular, $j^{narr}(\theta) = j^{br}(\theta)$ except, may be, for at most countable set $\Theta_f$ of values of $\theta$. It follows that broad and narrow solutions of (56) with the source term $f$ coincide, for $\theta \notin \Theta_f$. Furthermore, take a countable set $(f_i)_i \subset L^{((p_-)^*)'}(\Omega)$ dense in the $L^1(\Omega)$ topology. Then for all $\theta \notin \bigcup_i \Theta_i$, we have $u_{f_i,\theta}^{br} = u_{f_i,\theta}^{narr}$ for all $i \in \mathbb{N}$. Thanks to the $L^1$ contraction property (25) for the maps $f \mapsto u_{f,\theta}^{br}$ and $f \mapsto u_{f,\theta}^{narr}$, we conclude that

$$\left|\begin{array}{l} \text{in the case } p_\theta(x) = p(x) + \theta,\ \ p \in C(\overline{\Omega}; [p_-, p_+]), \\ \text{for all } |\theta| < p_- - 1 \text{ except, may be, for at most countable set of values of } \theta, \\ \text{the renormalized broad and narrow solutions to (56)} \\ \text{coincide for all } f \in L^1(\Omega). \end{array}\right. \tag{58}$$

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